Implicit Kinetic Schemes for Saint-Venant

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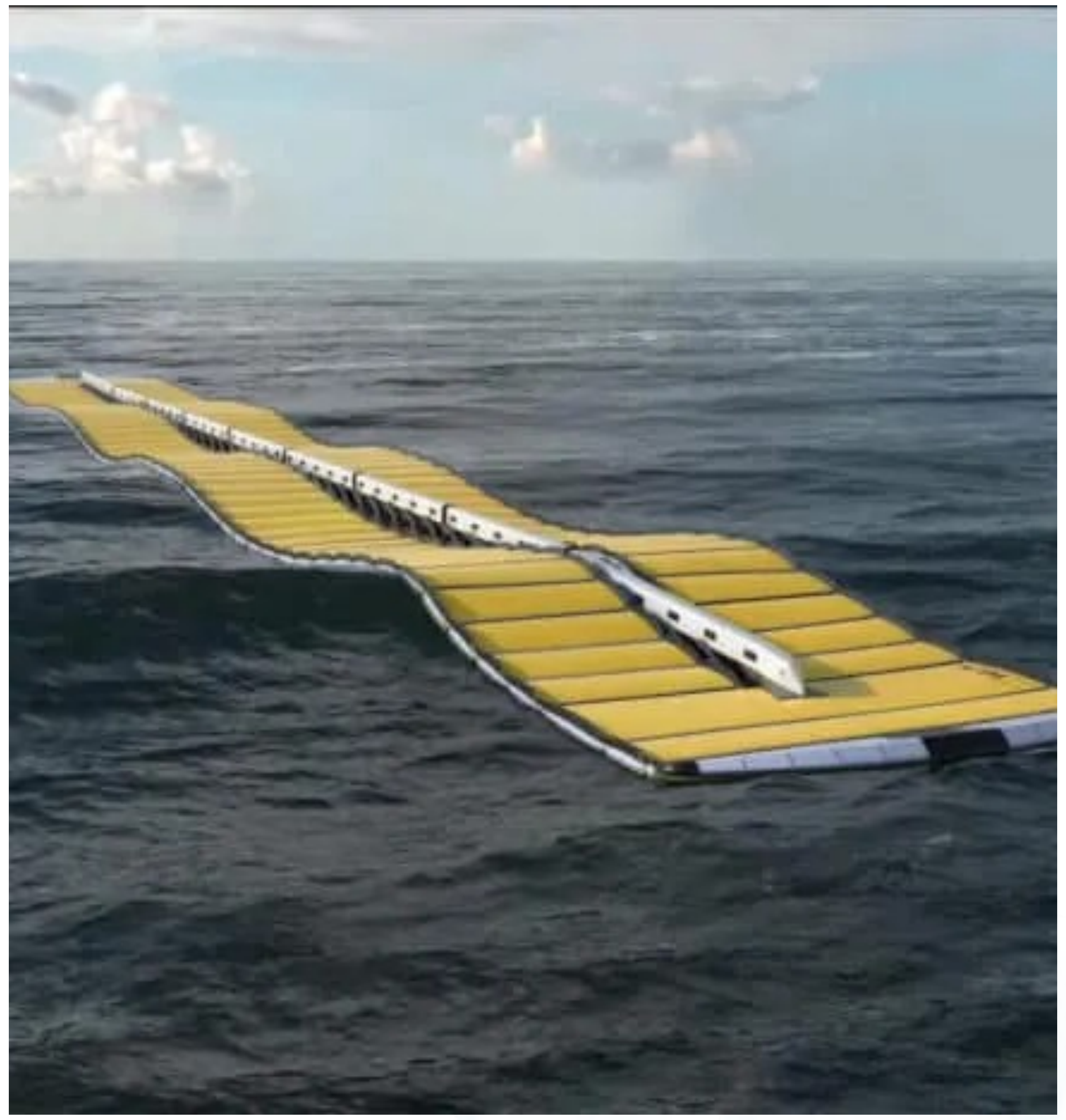
Geophysical flows

Why are we interested?



- water management (quality, availability)
- forecast natural disasters, mitigate their consequences
- energy production

1D Saint-Venant System

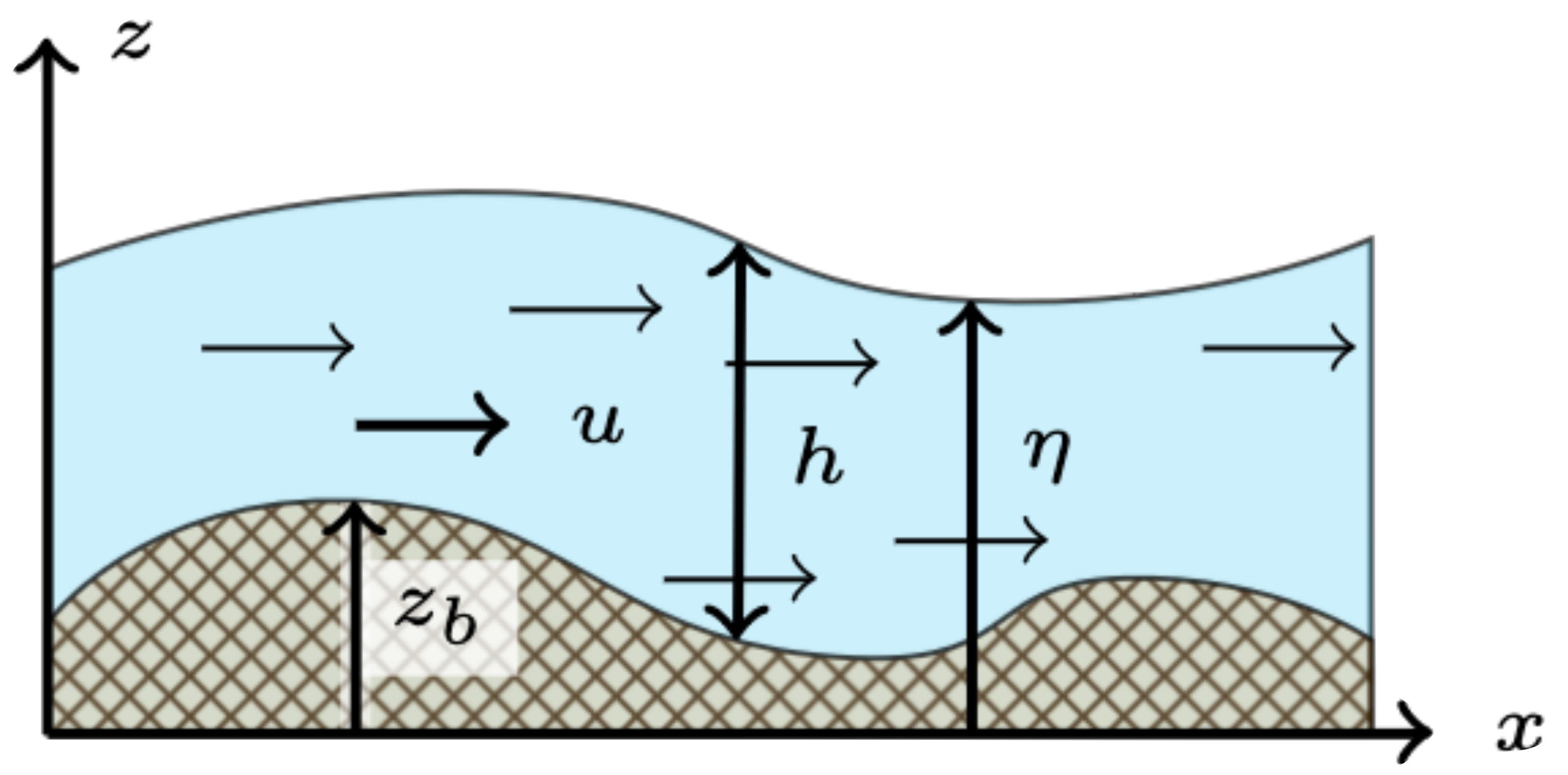
$$\frac{\partial h}{\partial t} + \partial_x(hu) = 0$$
$$\frac{\partial (hu)}{\partial t} + \partial_x(hu^2 + \frac{gh^2}{2}) = -gh\partial_x z_b$$

h water height

u averaged velocity

η free surface

z_b bottom topography



✦ vertical contribution of horizontal velocity is neglected

✦ widely studied (well-posedness, numerical approximation)

✦ hyperbolic structure

✓ implicit scheme using kinetic framework 1d,2d

✓ flat topography (explicit updates)

✓ variable topography (iterative strategy)

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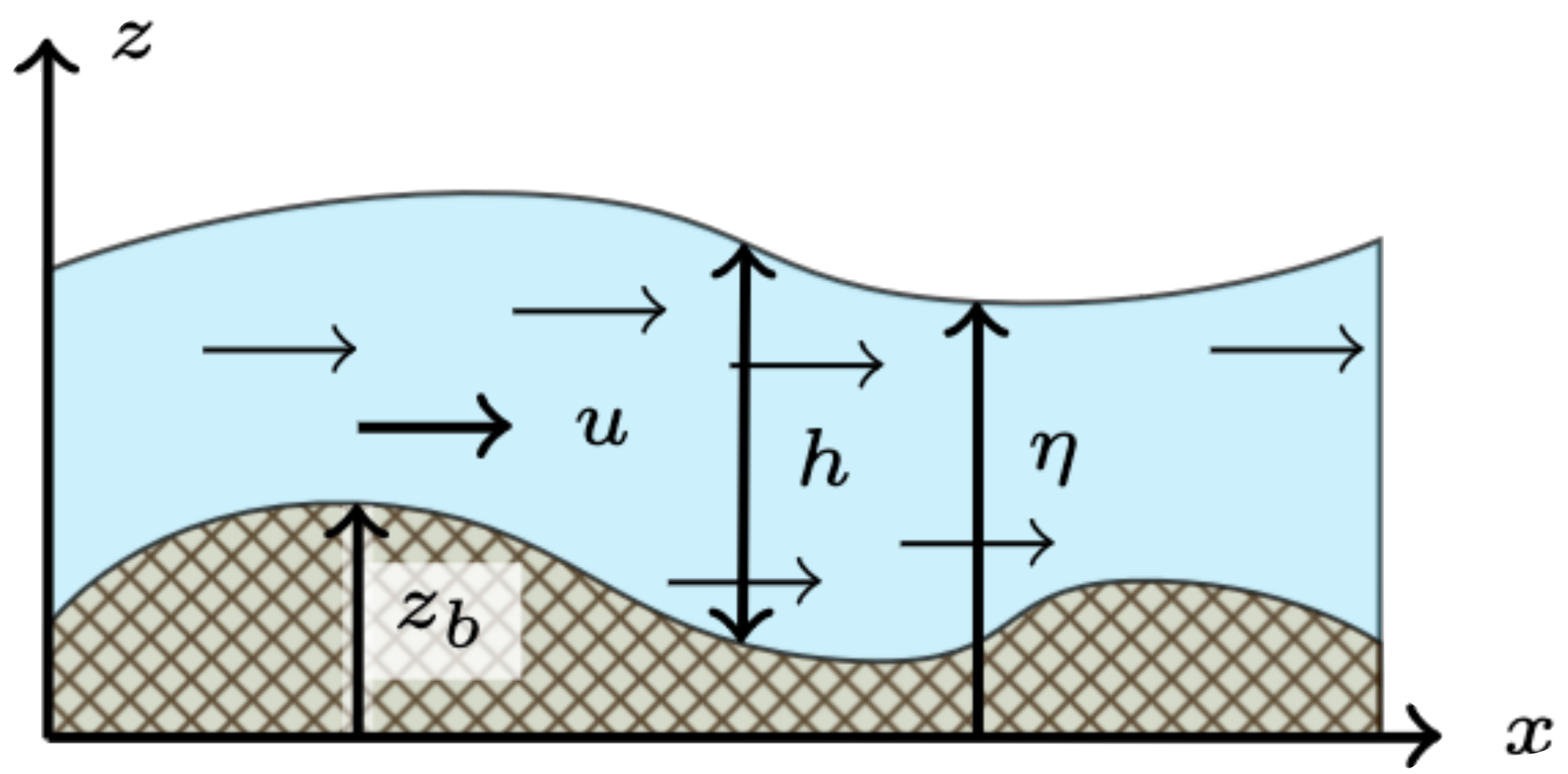

Properties

- Positivity. $h \geq 0$
- Stationary States $h + z_b \equiv Cst, \qquad u \equiv 0$
- Entropy inequality $\partial_t E(U) + \partial_x G(U) \leq 0, \qquad U = (h, hu)$

$$E(U) = \frac{hu^2}{2} + \frac{gh^2}{2} + ghz_b, \qquad G(U) = \left(\frac{hu^2}{2} + gh^2 + ghz_b\right) u$$

Kinetic Representation

Kinetic scale $f(t, x, \xi)$ distribution of particles with velocity $\xi \in \mathbb{R}$

Macroscopic scale h, hu recovered by integration $U_f = \int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} f(t, x, \xi) d\xi$

Boltzmann-type kinetic equations

$$\partial_t f(t, x, \xi) + \xi \partial_x f(t, x, \xi) - g(\partial_x z_b) \partial_\xi f(t, x, \xi) = \frac{1}{\varepsilon} \underbrace{Q[f](t, x, \xi)}_{\text{collisions}^*}$$

* $\int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} Q[f](t, x, \xi)\, d\xi = 0$

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Link to Saint-Venant Mesoscopic → Macroscopic

✦ Gibbs equilibrium $f \xrightarrow{\varepsilon \to 0} M(U_f, \xi)$

✦ Moment relations

$$\int_{\mathbb{R}} M(U, \xi) d\xi = h \qquad \int_{\mathbb{R}} \xi M(U, \xi) d\xi = hu \qquad \int_{\mathbb{R}} \xi^2 M(U, \xi) d\xi = hu^2 + g\frac{h^2}{2}$$

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BGK Splitting

- ✦ Transport Step $\partial_t f + \xi \partial_x f - g(\partial_x z_b) \partial_\xi f = 0$
- ✦ Relaxation Step $\partial_t f = \dfrac{M(U_f(t, x), \xi) - f(t, x, \xi)}{\varepsilon}$

Kinetic Representation

Moment Relations + Kinetic Entropy

$$\int_{\mathbb{R}} M(U,\xi)\, d\xi = h$$

$$\int_{\mathbb{R}} \xi M(U,\xi)\, d\xi = hu$$

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$$H(f,\xi) = \frac{\xi^2}{2}f + \frac{g^2\pi^2}{6}f^3 + gz_b f$$

Lemma. (Perthame and Simeoni 2001)

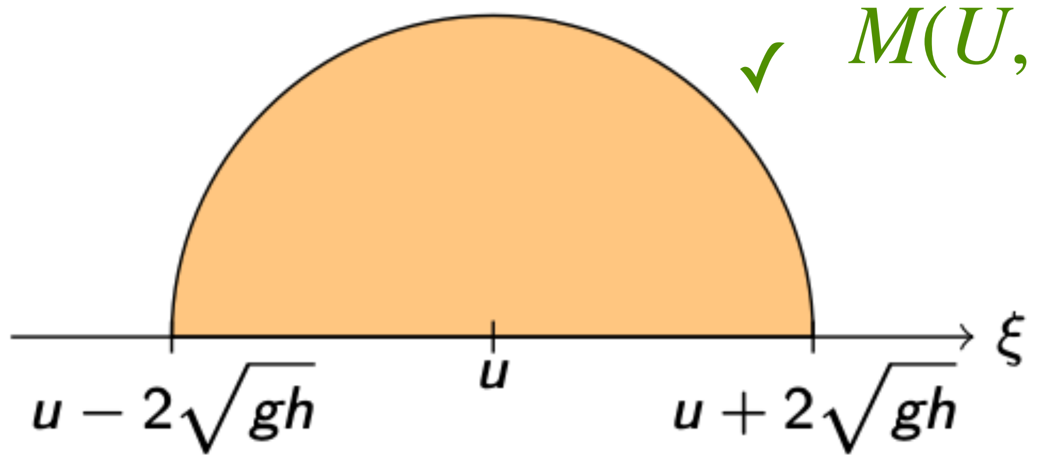
The function $H(f,\xi) = \frac{\xi^2}{2}f + \frac{g^2\pi^2}{6}f^3 + gz_b f$ is a kinetic entropy associated to the half-disk Maxwellian $M(U,\xi) = \frac{1}{g\pi}\left(2gh - (\xi-u)^2\right)_+^{1/2}$.

Kinetic Representation

The Maxwellian $M(U,\xi)$ minimizes the energy $\displaystyle\int_{\mathbb{R}} H(M(U_f,\xi),\xi)d\xi \leq \int_{\mathbb{R}} H(f,\xi)d\xi \quad \forall f \geq 0$

such that $\begin{pmatrix} h \\ hu \end{pmatrix} = \displaystyle\int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} f(t,x,\xi)d\xi.$

✓ $M(U,\xi) = \dfrac{1}{g\pi}\left(2gh - (\xi-u)^2\right)_+^{1/2}$



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Explicit Scheme (flat topography)

Consider the case of flat topography $z_b =$ Cst.

$$f_i^n(\xi) \approx \frac{1}{\Delta x} \int_{C_i} f(t^n, x, \xi) d\xi, \quad 1 \leq i \leq N$$

Explicit Kinetic Scheme

$$\frac{f_i^{n+1} - M_i^n}{\Delta t} + \frac{\xi}{\Delta x} \left(1_{\xi<0}(M_{i+1}^n - M_i^n) + 1_{\xi>0}(M_i^n - M_{i-1}^n) \right) = 0$$

Macroscopic Level

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + \frac{1}{\Delta x} \left(F(U_i^n, U_{i+1}^n) - F(U_{i-1}^n, U_i^n) \right) = 0$$

Proposition. (Audusse, Bouchut, Bristeau, Sainte-Marie 2016)

Under the CFL $\sigma|\xi| \leq 1$, the explicit kinetic scheme satisfies $h_i^{n+1} \geq 0$ and the following entropy inequality

$$\eta(U_i^{n+1}) \leq \eta(U_i^n) - \sigma(G_{i+1/2}^n - G_{i-1/2}^n), \qquad \sigma = \Delta t / \Delta x.$$

Implicit Scheme (flat topography)

Implicit Kinetic Scheme

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + \frac{\xi}{\Delta x}\left(1_{\xi<0}(f_{i+1}^{n+1} - f_i^{n+1}) + 1_{\xi>0}(f_i^{n+1} - f_{i-1}^{n+1})\right) = 0$$

Matrix Form $(\mathbf{I} + \sigma\mathbf{L})f^{n+1} = M + \sigma B^n$

- ✓ $(\mathbf{I} + \sigma\mathbf{L})$ invertible with positive coefficients
- ✓ System has unique solution with positive quantities

Proposition. (Rigal, Sainte-Marie, E.H. 2023)

$\forall \Delta t > 0$ the implicit kinetic scheme satisfies $h_i^{n+1} \geq 0$ and the following entropy inequality

$$\eta(U_i^{n+1}) \leq \eta(U_i^n) - \sigma(G_{i+1/2}^{n+1} - G_{i-1/2}^{n+1}) + D_i^n, \quad D_i^n \leq 0.$$

Implicit Scheme (flat topography)

Implicit Kinetic Scheme

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Matrix Form $(\mathbf{I} + \sigma\mathbf{L})f^{n+1} = M + \sigma B^n$

✓ $(\mathbf{I} + \sigma\mathbf{L})$ invertible with positive coefficients
✓ System has unique solution with positive quantities
✓ Positive water height
✓ Entropy inequality

✕ Macroscopic Updates (half-disk Maxwellian)

$$1_{\pm\xi>0}\begin{pmatrix}1\\ \xi\\ \xi^2\end{pmatrix}\frac{(\pm\sigma\xi)^k}{(1\pm\sigma\xi)^{k+1}}M(U,\xi), \qquad 0 \le k \le P-1$$

Implicit Scheme (flat topography)

Implicit Kinetic Scheme

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + \frac{\xi}{\Delta x}\left(1_{\xi<0}(f_{i+1}^{n+1} - f_i^{n+1}) + 1_{\xi>0}(f_i^{n+1} - f_{i-1}^{n+1})\right) = 0$$

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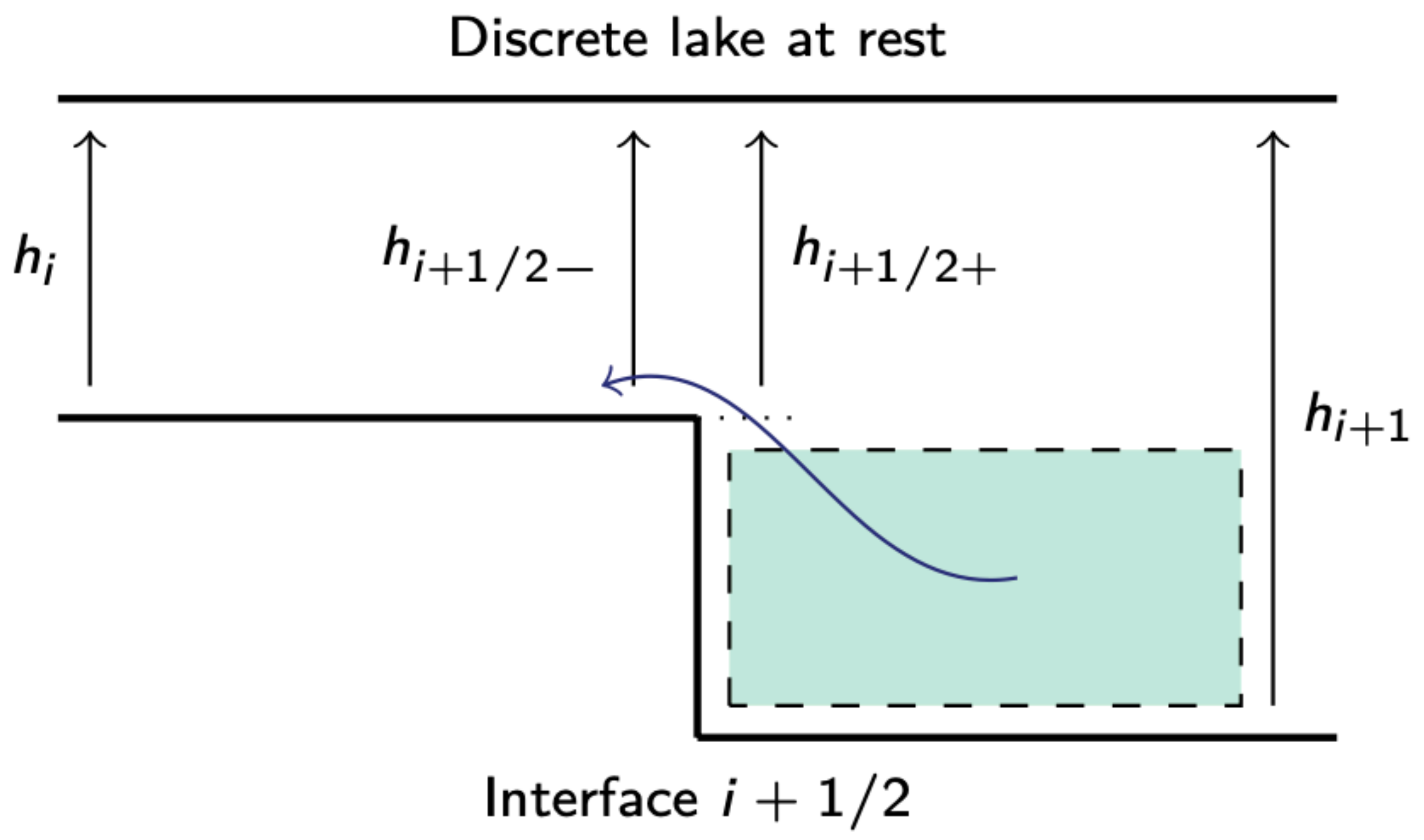
<u>Possible Solutions</u>

- ✦ Quadratures
- ✦ Another Maxwellian $M(U,\xi) = \frac{h}{c}\chi\left(\frac{\xi - u}{c}\right)$
- ✦ Iterative Methods

Case of Variable Topography

✓ **Audusse, Bouchut, Bristeau, Klein, et al. 2004** "A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows"

Hydrostatic reconstruction



Explicit scheme with hydrostatic reconstruction

$$\frac{f_i^{n+1} - M_i^n}{\Delta t} + \frac{\xi}{\Delta x}\left(M_{i+1/2}^n - M_{i-1/2}^n\right) + \frac{1}{\Delta x}(\xi - u_i^n)\left[M_{i-1/2+}^n - M_{i+1/2-}^n\right] = 0$$

Proposition. (Audusse, Bouchut, Bristeau, Sainte-Marie 2016)
The explicit scheme (variable topography) admits an entropy inequality of the form

$$\eta(U_i^{n+1}, z_i) \leq \eta(U_i^n, z_i) - \sigma(G_{i+1/2}^n - G_{i-1/2}^n) + D_i^n + \varepsilon_i^n, \quad \varepsilon_i^n \geq 0$$

ε_i^n quadratic error that vanishes as $u_i^n \to 0$.

✓ Preserves lake at rest
✓ Balances pressure variation with source term

Implicit Scheme (variable topography)

Implicit scheme with hydrostatic reconstruction

$$\frac{f_i^{n+1} - M_i^n}{\Delta t} + \frac{\xi}{\Delta x}\left(M_{i+1/2}^{n+1} - M_{i-1/2}^{n+1}\right) + \frac{1}{\Delta x}(\xi - u_i^{n+1})\left[M_{i-1/2+}^{n+1} - M_{i+1/2-}^{n+1}\right] = 0$$

Non-linear implicit update cannot be computed exactly $\Rightarrow$ Use iterative process (Gauss-Jacobi)

$$(1+\alpha)f_i^{n+1,k+1} = M_i + \alpha M_i^{n+1,k} - \sigma^k \xi \left(M_{i+1/2}^{n+1,k} - M_{i-1/2}^{n+1,k}\right) + \sigma^k(\xi - u_i^{n+1,k})\left[M_{i-1/2+}^{n+1,k} - M_{i+1/2-}^{n+1,k}\right]$$

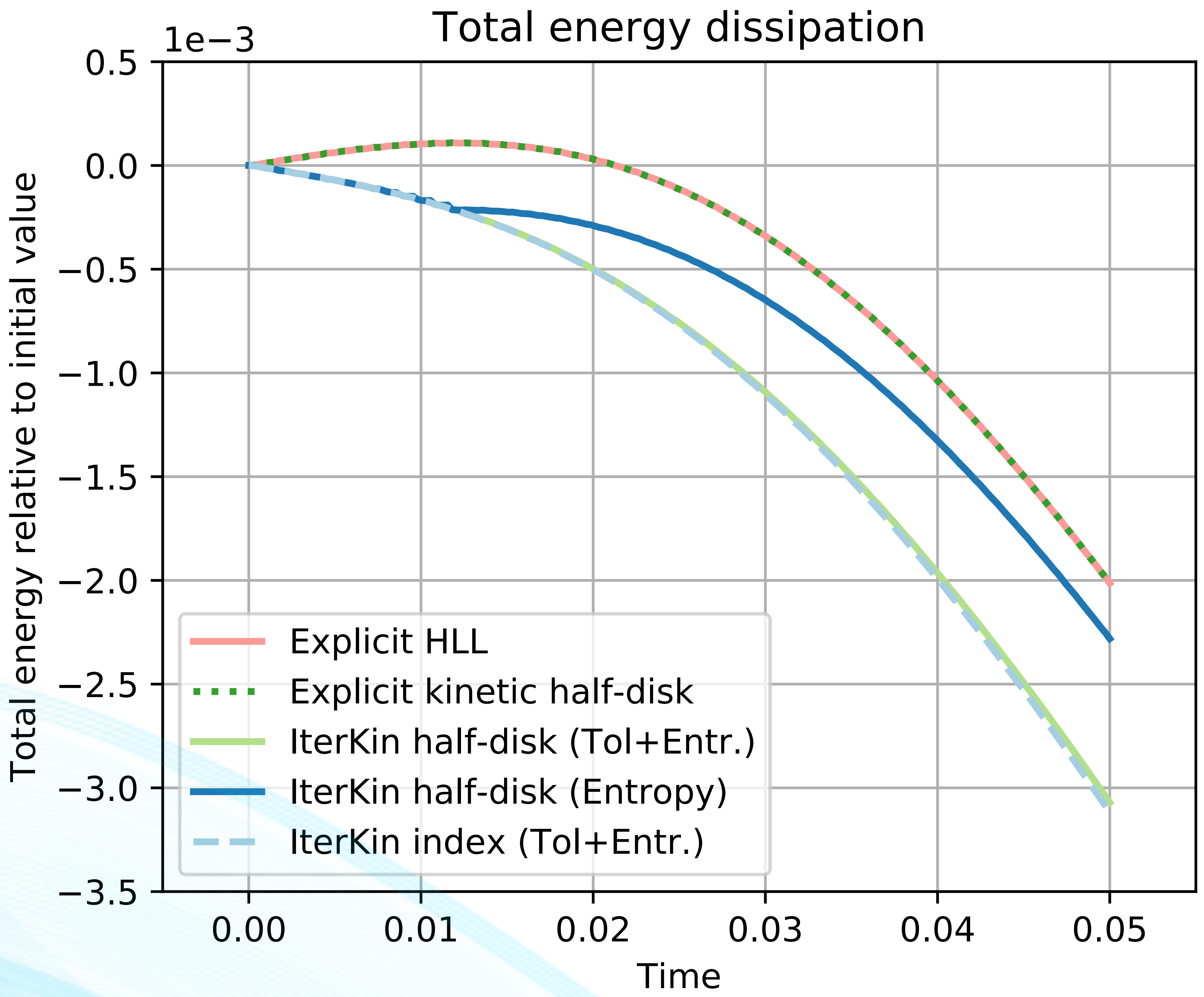
Proposition. (Rigal, Sainte-Marie, E.H. 2023)

✓ $h_i^{n+1,k+1} \geq 0$ if $h^n, h^{n+1,k} \geq 0$ and $\forall \xi \in \mathbb{R} \quad (\sigma^k |\xi| - \alpha) M_i^{n+1,k} \leq M_i^n$

✓ Iterative process satisfies $\eta(U_i^{n+1,k+1}, z_i) \leq \eta(U_i^{n,k}, z_i) - \sigma^k(G_{i+1/2}^{n+1,k} - G_{i-1/2}^{n+1,k}) + D_i^{n+1,k}, \quad D_i^{n+1,k} \leq 0$ from some rank.

Test Case.

- ✦ Periodic boundary conditions
- ✦ $h + z_b = Cst$
- ✦ $u = Cst \neq 0$



2D - Test case.

- ✦ Analytic solution
- ✦ Difficult to capture