Unfitted HHO method with polynomial extension for elliptic interface problems

Romain Mottier[§†‡]

Alexandre Ern[†‡] and Erik Burman[*]

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[§] CEA, DAM, DIF, F-91297 Arpajon, France
[‡] CERMICS, Ecole des Ponts, F-77455 Marne la Vallée cedex 2
[†] SERENA Project-Team, INRIA Paris, F-75589 Paris France
[*] Department of Mathematics, University College London, London, UK–WC1E 6BT, UK

Email adress: romain.mottier@outlook.fr

Table of Contents

Table of Contents

Domain decomposition

- **Domain Ω:** $\overline{\Omega} := \overline{\Omega_1} \cup \overline{\Omega_2}$
- **Interface Γ:** $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$
- **Jump across Γ:** $[\![u]\!]_\Gamma := u_{|\Omega_1} - u_{|\Omega_2}$

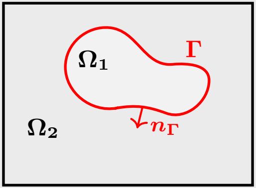

Fig. 1: Model problem

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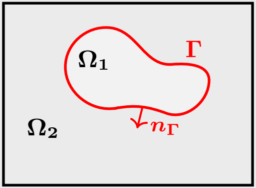

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Elliptic interface problem

- **Strong form:** Find $u \in H^1(\Omega_1 \cup \Omega_2)$ such that

$$\begin{cases} -\nabla \cdot (\kappa \nabla u) = f & \text{in } \Omega_1 \cup \Omega_2 \\ [\![u]\!]_\Gamma = g_D & \text{on } \Gamma \\ [\![\kappa \nabla u]\!]_\Gamma \cdot \boldsymbol{n}_\Gamma = g_N & \text{on } \Gamma \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Unfitted methods

- **Minimize complexity of mesh generation**
- **Handle cut cells by doubling unknowns**
- **Need to integrate polynomials in cut cells** (e.g. by submeshing)
- **Price to pay : Need to stabilize ill-cut cells**

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Unfitted FEM methods

- Introduced by [Hansbo and Hansbo, 2002]
- Standart technique for stabilization: **Ghost penalty** [Burman, 2010]

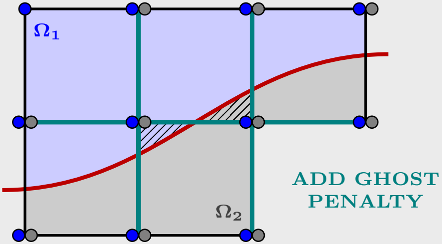

Fig. 2: Doubling of Q_1-FEM unknowns, ill-cut cells (dashes) and set of ghost-penalty faces

Fitted HHO methods

- **Seminal papers:** [Di Pietro, Ern, and Lemaire, 2014], [Di Pietro and Ern, 2015]
- **Main features:**
  - Design based on cell and face unknowns
  - General meshes: polyhedral meshes, hanging nodes
  - Attractive computational cost: Static condensation
  - Local conservativity at the cell level

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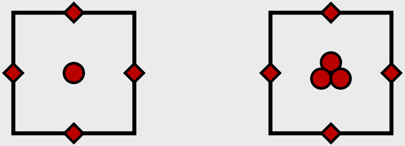
Unfitted HHO methods

- **Seminal papers:** [Burman and Ern, 2018] [Burman, Cicuttin, Delay, and Ern, 2021]
- **Main features:**
  - Doubling of cell and face unknowns in cut cells
  - Cut stabilization by cell agglomeration
- **New approach for cut stabilization: Polynomial extension**
  - Use of similar technique for unfitted FEM [Badia, Verdugo, and Martín, 2018]

Table of Contents

Degrees of freedom

- **Polynomial unknowns attached to mesh cells and faces**



HHO unknowns:

$$\hat{\boldsymbol{u}}_h := (\boldsymbol{u}_{\mathcal{T}}, \boldsymbol{u}_{\mathcal{F}}) \in \hat{\mathcal{U}}_h$$

Cell unknowns, degree $\boldsymbol{k}' \in \{\boldsymbol{k}, \boldsymbol{k}+1\}$ **Face unknowns, degree $\boldsymbol{k} \geq 0$**

Fig. 3: Local HHO unknowns. **Left:** $k' = k = 0$. **Right:** $k' = k + 1 = 1$.

- Equal-order: $k' = k$
- Mixed-order: $k' = k + 1$

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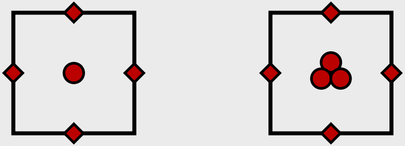

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Global degrees of freedom

- Mesh $\mathcal{T}_h$ with faces $\mathcal{F}_h$
- **Global HHO spaces:** $$\widehat{\mathcal{U}}_h := \bigtimes_{T \in \mathcal{T}_h} \mathbb{P}^{k'}(T;\mathbb{R}) \times \bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^{k}(F;\mathbb{R})$$

Design of the local gradient reconstruction operator

- **Gradient reconstruction operator:**
  - $(\boldsymbol{\nabla u})_{|T} \to \mathbf{G}_{\boldsymbol{T}}(\hat{\boldsymbol{u}}_T) \in \mathbb{P}^k(T;\mathbb{R}^d)$

    Design of $\mathbf{G}_{\boldsymbol{T}}(\hat{\boldsymbol{u}}_T)$ mimics an integration by parts

$$(\boldsymbol{G}_T(\hat{u}_T), \boldsymbol{q})_T = (\nabla u_T, \boldsymbol{q})_T - (u_T - u_{\partial T}, \boldsymbol{q} \cdot \boldsymbol{n}_T)_{\partial T}, \quad \forall \boldsymbol{q} \in \mathbb{P}^k(T;\mathbb{R}^d)$$

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Design of the local stabilization operator

- **Stabilization operator:** $\boldsymbol{\delta}_{\partial T}(\hat{\boldsymbol{u}}_T) := \boldsymbol{u}_{\partial T} - \boldsymbol{u}_{T|\partial T} \approx \mathbf{0}$

  Matching of cell dofs trace with face dofs (weakly)

  - **Equal-order discretization:** Specific stabilization to HHO (not used in unfitted HHO)
  - **Mixed-order discretization:** Same as HDG (Lehrenfeld-Schöberl)

$$s_T(\hat{u}_T, \hat{w}_T) := \kappa\, h_T^{-1}(\Pi^k_{\partial T}(u_T - u_{\partial T}), w_T - w_{\partial T})_{\partial T}$$

Main advantages of HHO methods

- **Improved error estimates for smooth solutions:**
  - H^1-error: $\mathcal{O}\left(h^{k+1}\right)$
  - L^2-error: $\mathcal{O}\left(h^{k+2}\right)$

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- **Attractive computational costs:** Elimination of cell unknowns by Schur complement (static condensation) :
  - Global problem couples only face dofs
  - Cell dofs recovered by local post-processing

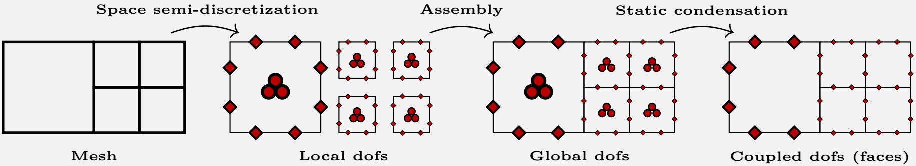

Fig. 4: Assembly and Schur complement procedure in the framework of HHO schemes

Table of Contents

Unfitted meshes and local unknowns

- **Mesh partitioning:** $\mathcal{T}_h := \mathcal{T}_h^\circ \cup \mathcal{T}_h^{\mathrm{OK}} \cup \mathcal{T}_h^{\mathrm{KO}}$
  - $\mathcal{T}_h^{\mathrm{KO},1} \cup \mathcal{T}_h^{\mathrm{KO},2} = \emptyset$ if mesh fine enough [Burman and Ern, 2018]

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- **Doubling local unknowns in cut cells:**

$$\hat{u}_T := (\hat{u}_{T^1}, \hat{u}_{T^2}) := (u_{T^1}, u_{(\partial T)^1}, u_{T^2}, u_{(\partial T)^2}) \in \widehat{\mathcal{U}}_T := \widehat{\mathcal{U}}_{T^1} \times \widehat{\mathcal{U}}_{T^2}$$

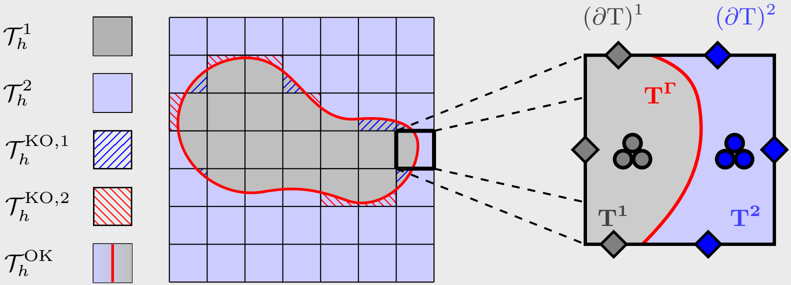


Fig. 5: Left. Types of cells involved in unfitted meshes.
Right. Local dofs in cut cell.

Pairing operator

$$\mathcal{N}_i : \mathcal{T}_h^{\mathrm{KO},i} \ni S \longmapsto T \in (\mathcal{T}_h^i \cup \mathcal{T}_h^{\mathrm{OK}} \cup \mathcal{T}_h^{\mathrm{KO},\bar{\imath}}) \cap \Delta_1(S), \quad \forall i \in \{1,2\}$$

- $\Delta_1(S)$: first layer of neighboring cells of S

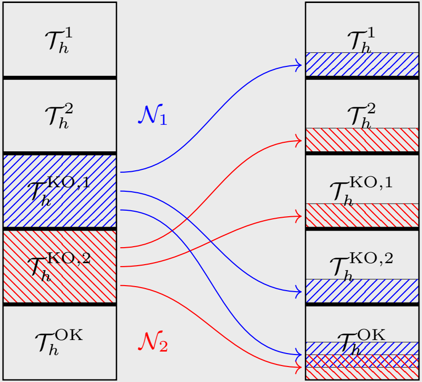


Fig. 6: Pairing of ill-cut cells

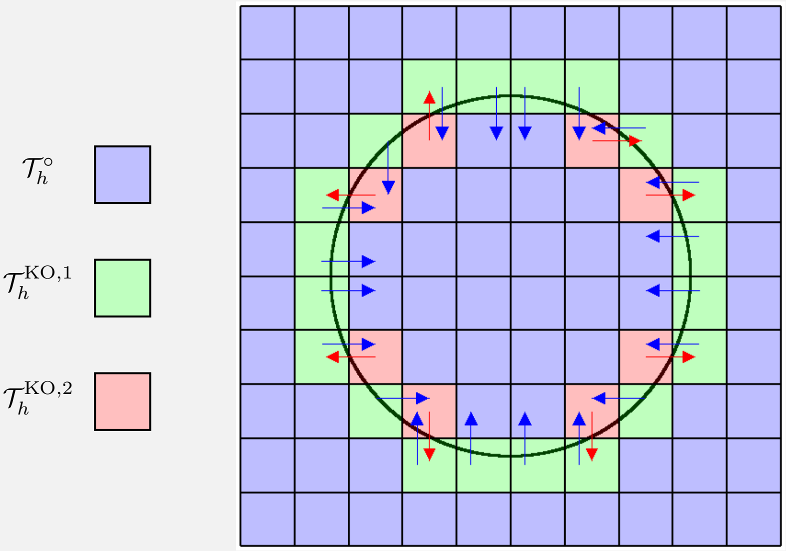


Fig. 7: Exemple of pairing procedure for coarse Cartesian mesh cut by circular interface

Cell agglomeration vs. Polynomial extension

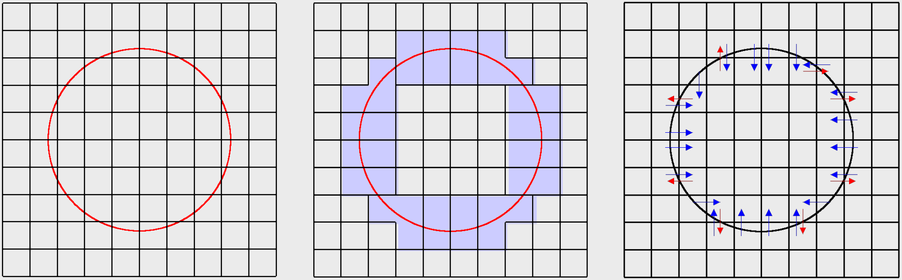
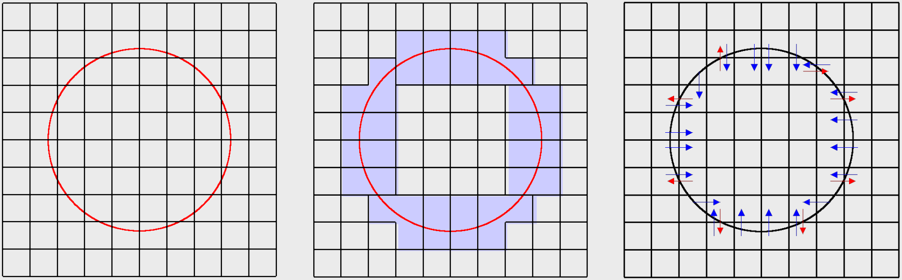


Fig. 8: **Left.** Initial mesh with circular interface. **Middle.** Cell agglomeration. **Right.** Stencil modification for polynomial extension

Cell agglomeration vs. Polynomial extension



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- **Cell agglomeration:**
  - ✓ Leverages on polyhedral capacity of HHO methods
  - ✗ Intrusive on mesh data structure

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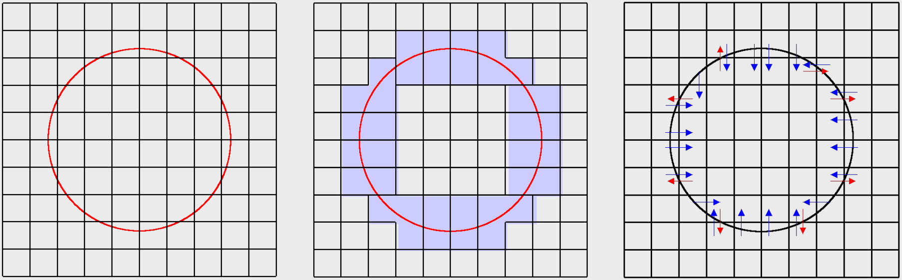

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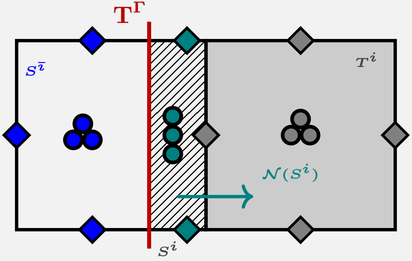
- **Cell agglomeration:**
  - ✓ Leverages on polyhedral capacity of HHO methods
  - ✗ Intrusive on mesh data structure
- **Polynomial extension:**
  - ✓ Works on initial mesh (non-intrusive)
  - ✗ Requires modification of the stencil (intrusive at the assembly level)

Table of Contents

- **UNCUT CELLS:** $T \in \mathcal{T}_h^i$
  - Stencil includes dofs of ill-cut cell(s)

$$\hat{u}_T^+ := (\hat{u}_T, (\hat{u}_S)_{S \in \mathcal{N}^{-1}(T)})$$



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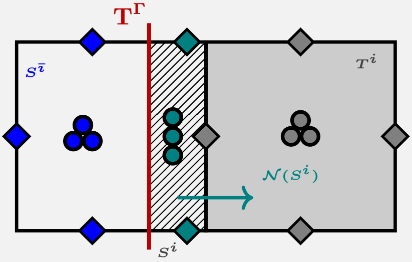

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Design of the local gradient reconstruction in the uncut cells

- **Classical gradient reconstruction:**

$$(\boldsymbol{G}_T(\hat{u}_T), \boldsymbol{q})_T = (\nabla u_T, \boldsymbol{q})_T - (u_T - u_{\partial T}, \boldsymbol{q} \cdot \boldsymbol{n}_T)_{\partial T}$$

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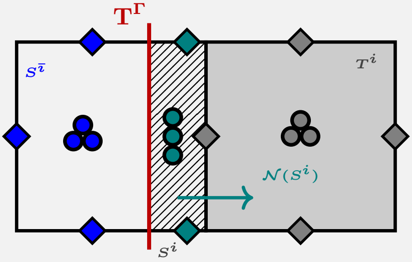

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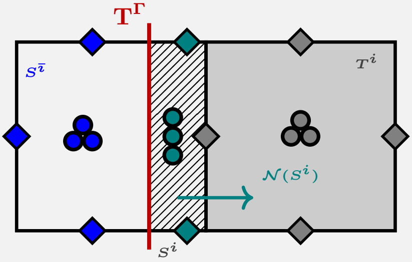

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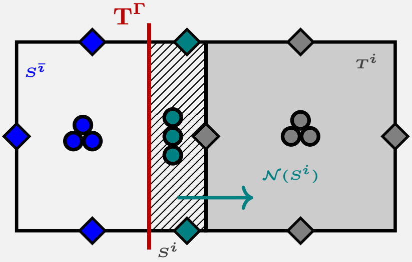

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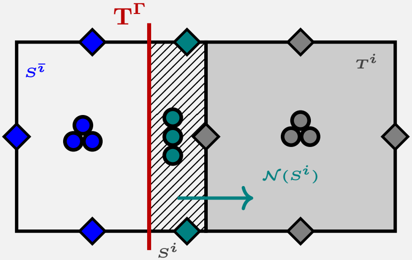

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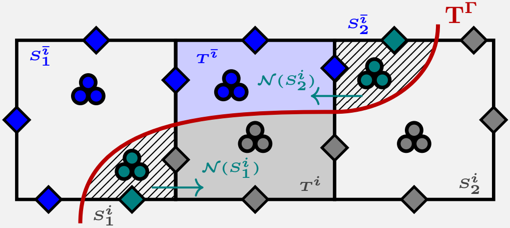

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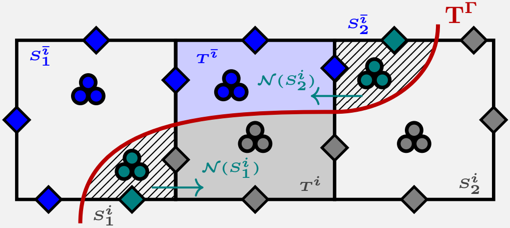

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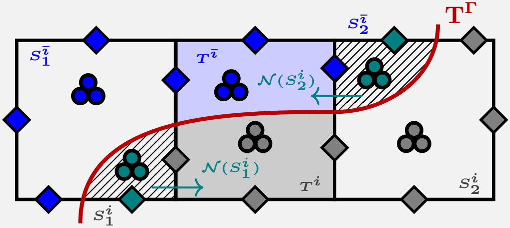

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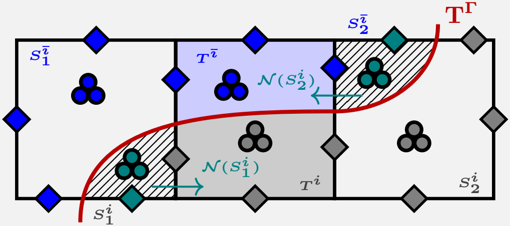

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- **ILL-CUT CELLS:** $T \in \mathcal{T}_h^{\mathbf{KO},i}$
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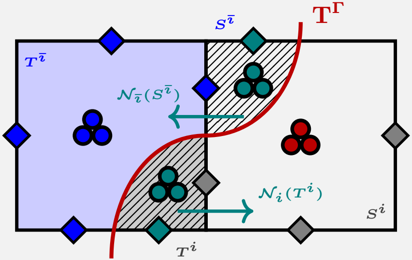

Fig. 11: Pairing configuration

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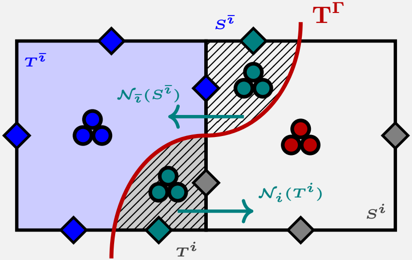

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Design of the local gradient reconstruction in the ill-cut cells

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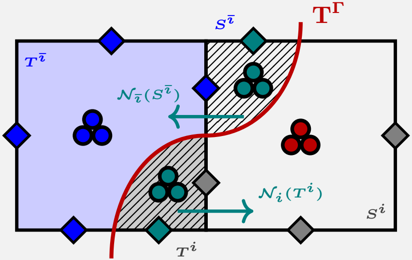

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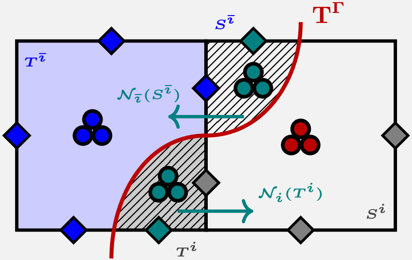

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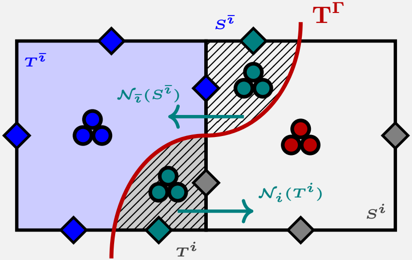

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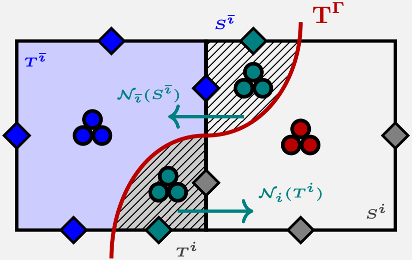

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HHO stabilization

- **Classical HHO stabilization:** $\forall i \in \{1,2\}$,

$$s_{T^i}(\hat{u}_{T^i}, \hat{w}_{T^i}) := \kappa_i h_T^{-1} (\Pi^k_{(\partial T)^i}(u_{T^i} - u_{(\partial T)^i}), w_{T^i} - w_{(\partial T)^i})_{(\partial T)^i}$$

- **Stabilization with polynomial extension (e.g. $T \in \mathcal{T}_h^{\mathbf{OK}}$):** $\forall i \in \{1,2\}$,

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Design of the cut stabilization operator (Nitsche's term)

- **Classical cut stabilization operator:** $\forall i \in \{1,2\}$,

$$s^\Gamma_{T^i}(\hat{u}_T^+, \hat{w}_T^+) := \delta_{i1} \kappa_1 h_T^{-1} (\llbracket u_T \rrbracket_\Gamma, \llbracket w_T \rrbracket_\Gamma)_{T^\Gamma}$$

- **Cut stabilization with polynomial extension (e.g. $T \in \mathcal{T}_h^{\mathbf{OK}}$):** $\forall i \in \{1,2\}$,

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Table of Contents

Global discrete problem

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- $\ell_h(\hat{w}_h) := \sum_{T \in \mathcal{T}_h^{\circ}} (f, w_{T^i})_{T^i}$
  $+ \sum_{T \in \mathcal{T}_h^{\mathrm{KO}}} \left\{ (f, w_{\mathcal{N}(T)^i})_{T^i} + (f, w_{T^{\bar{\imath}}})_{T^{\bar{\imath}}} \right\} + \sum_{T \in \mathcal{T}_h^{\mathrm{OK}}} \sum_{i \in \{1,2\}} (f, w_{T^i})_{T^i}$
  - For simplicity, we consider $g_D = g_N = 0$

Algebraic realization for gradient reconstruction

- Algebraic realization of $(\boldsymbol{G}^k_{T^i}(\hat{u}_T^+), \boldsymbol{G}^k_{T^i}(\hat{w}_T^+))_{T^i}$ (e.g. $\forall T \in \mathcal{T}_h^{\mathrm{OK}}$):

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  - **Extension of local bilinear form $\longrightarrow$ Modification of assembly**

Error analysis

- Based on [Burman, Cicuttin, Delay, and Ern, 2021]
  - Stability (coercivity)
  - Consistency
  - Quasi-optimal error estimates
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Thank you for your attention !