Conductivity problems in random media

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Joint work with Lihan WANG (CMU)

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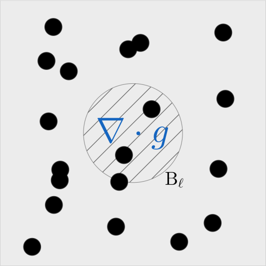
Conductivity in random media



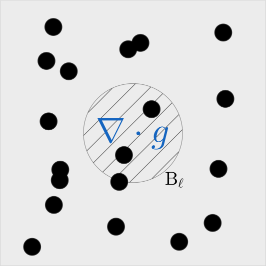
Goal: Compute the electric field generated by a localized charge in random heterogeneous media

Model: Compute ∇u where $u : \mathbb{R}^3 \to \mathbb{R}$ solves

$$-\nabla \cdot a\nabla u = \nabla \cdot g \quad \text{in } \mathbb{R}^3 \quad \text{such that } u(x) \underset{|x|\uparrow\infty}{\to} 0,$$

with $a = a(x)$ the (heterogeneous) conductivity such that $\lambda|\xi|^2 \leq a\xi \cdot \xi \leq |\xi|^2$

Conductivity in random media



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Random heterogeneous media: a is sampled from a probability measure $\mathbb{P}$

- **Stationarity**, i.e $a(\cdot + x)$ and a have the same distribution for all $x \in \mathbb{R}^3$

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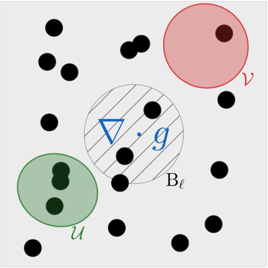

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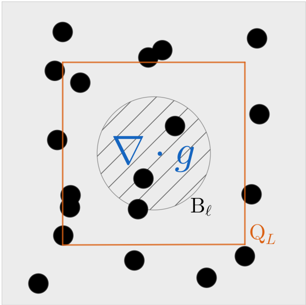
Random heterogeneous media: a is sampled from a probability measure $\mathbb{P}$

- **Stationarity**, i.e $a(\cdot + x)$ and a have the same distribution for all $x \in \mathbb{R}^3$
- **Ergodicity**, i.e $a|_{\mathcal{U}}$ and $a|_{\mathcal{V}}$ decorrelates as $\text{dist}(\mathcal{U}, \mathcal{V}) \uparrow \infty$

Computation of the electric field



In practice: We use a finite volume approximation

$$\begin{cases} -\nabla \cdot a \nabla u_L = \nabla \cdot g & \text{in } Q_L, \\ u_L = u_{\mathrm{bc}} & \text{on } \partial Q_L. \end{cases}$$

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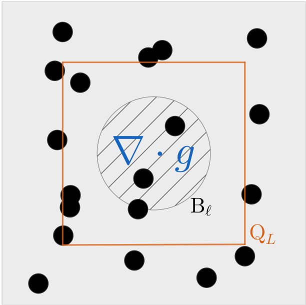

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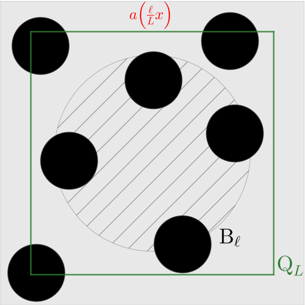
Question: Construct u_{bc} that leads to optimal errors in L and ℓ !

Construction of the boundary condition

I. Large-scale behaviour in random media: Papanicolaou, Varadhan, Dal Maso, Modica; Gloria, Neukamm, Otto, Armstrong, Kuusi, Mourrat, Smart



$$L \sim \ell$$

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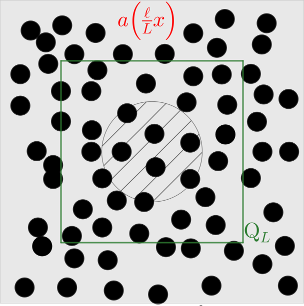

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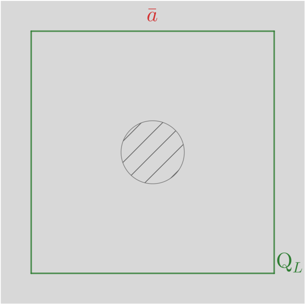

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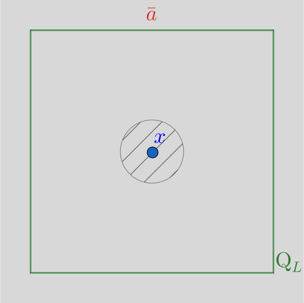

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The media homogenizes and is described by its two-scale expansion

$$-\nabla \cdot a \nabla G(x, \cdot) = \delta(x - \cdot), \quad G(x, y) \approx \underbrace{(1 + \phi_i^{(1)}(x)\partial_{x_i})(1 + \phi_i^{(1)}(y)\partial_{y_i})}_{\text{Two-scale expansion}} \bar{G}(x - y)$$

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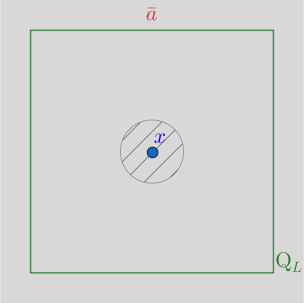

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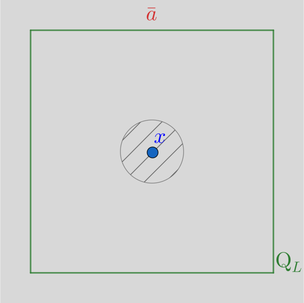

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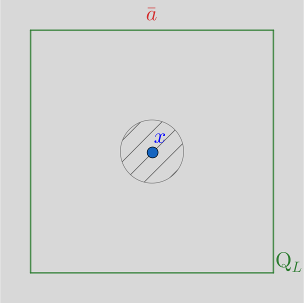

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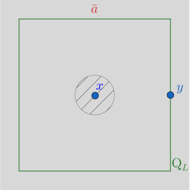

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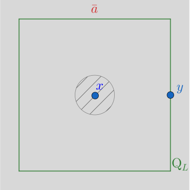

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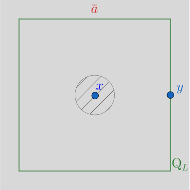

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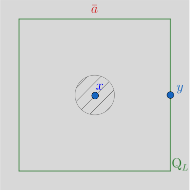

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$$u(x) \approx (1+\phi_i^{(1)}(x)\partial_{x_i})\Bigg(\underbrace{\int_{\mathbb{R}^d} \bar{G}(x-y)\nabla \cdot g(y)\mathrm{d}y}_{:=\bar{u}} - \int_{\mathbb{R}^d} \phi_i^{(1)}(y)\partial_i \bar{G}(x-y)\nabla \cdot g(y)\mathrm{d}y\Bigg).$$

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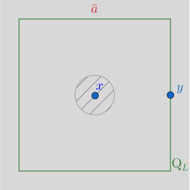

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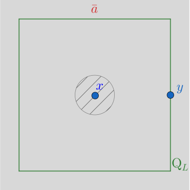

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Ansatz:

$$\boxed{u_{\mathrm{bc}} = (1+\phi_i^{(1)}\partial_i)\Big(\bar{u} - \partial_i \bar{G}(x)\int \phi_i^{(1)}(y)\nabla \cdot g(y)\mathrm{d}y\Big)}$$

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There exists $\phi^{(1)} \in \mathrm{H}^1_{\mathrm{loc}}$ *and*

$$\mathbb{E}\left[|\phi^{(1)}(x)|^2\right]^{\frac{1}{2}} \lesssim \begin{cases} (1+|x|)^{1-\frac{\beta}{2}} & \text{for } \beta < 2 \\ \log^{\frac{1}{2}}(2+|x|) & \text{for } \beta = 2 \\ 1 & \text{for } \beta > 2 \end{cases}$$

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For $\beta > 2$, we can go to a second-order approximation ! Fischer, Bella, Fehrman, Otto ('17)

A second-order expansion

Second-order correctors: Solution $\phi_{ij}^{(2)}$ of

$$-\nabla \cdot a\nabla\phi_{ij}^{(2)} = \nabla \cdot a\phi_i^{(1)} e_j + e_j \cdot (a(e_i + \nabla\phi_i^{(1)}) - \bar{a}e_i) \text{ in } \mathbb{R}^3$$

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Ansatz:

$$u_{\mathrm{bc}} = \underbrace{(1 + \phi^{(1)}_i\partial_i + \phi^{(2)}_{ij}\partial_{ij})}_{2^{nd}\text{-order two-scale expansion}} \left(\bar{u} - \underbrace{\partial_i \bar{G}(x)\int \phi^{(1)}_i(y)\nabla\cdot g(y)\mathrm{d}y}_{\text{Dipole correction}} + \underbrace{\partial_{ij}\bar{G}(x)c_{ij}}_{\text{Quadrupole correction}}\right)$$

The rigorous result

Theorem (C.-Wang ('24))

There exists $C, \gamma > 0$ such that for any $R \leq L$ and $L \geq C\ell$

$$\mathbb{P}\left(\left(\fint_{\mathrm{B}_R} |\nabla u - \nabla u_L|^2\right)^{\frac{1}{2}} \leq C \underbrace{\left(\frac{\ell}{L}\right)^d}_{\text{Dipole scaling}} \underbrace{L^{-\frac{\beta\wedge 3}{2}}}_{\text{Fluctuation scaling}}\right) \geq 1 - C\exp\left(-\ell^\gamma - R^\gamma - L^{\frac{1}{C}}\right)$$

Extend results from Jianfeng, Otto and Wang ('21) for the case $\beta \gg 1$

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How do we compute the boundary condition: There quantities to compute

$$\phi^{(1)}, \quad \phi^{(2)} \quad \text{and} \quad \bar{a}e_i := \mathbb{E}\left[a(e_i + \nabla\phi_i^{(1)})\right]$$

$$-\nabla\cdot a\nabla\phi_i^{(1)} = \nabla\cdot ae_i \text{ in } \mathbb{R}^3$$

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Approximation of the correctors

$$\bar{a}e_i = \mathbb{E}\Big[a(e_i + \nabla\phi_i^{(1)})\Big] = \lim_{L\uparrow\infty} \fint_{Q_L} a(e_i + \nabla\phi_i^{(1)})$$

We solve

$$\frac{1}{T}\phi_{i,L}^{(1)} - \nabla\cdot a\nabla\phi_{i,L}^{(1)} = \nabla\cdot ae_i \text{ in } Q_L, \quad \phi_{i,L}^{(1)} = 0 \text{ on } \partial Q_L$$

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Theorem (C.-Wang ('24))

For any $p < \infty$

$$\mathbb{E}\Big[\fint_{\mathrm{Q}_L} |(\phi_L^{(1)} - \phi^{(1)}, \sqrt{T}(\nabla\phi_L^{(1)} - \nabla\phi^{(1)}), \nabla\phi_L^{(2)} - \nabla\phi^{(2)})|^2\Big]^{\frac{1}{2}} \lesssim T^{\frac{1}{2}-\frac{\beta\wedge 3}{4}} + \Big(\frac{\sqrt{T}}{L}\Big)^p$$

References

The paper: C., Wang. *Artificial boundary conditions for random ellitpic systems with correlated coefficient field*, Multiscale modelling and simulations ('24)

Related works:

- Jianfeng, Otto, Wang. *Optimal artificial boundary conditions based on second-order correctors for three dimensional random elliptic media*, Preprint ('21)
- C. *Optimal decay of the parabolic semigroup in stochastic homogenization for correlated coefficient fields*, Stochastics and Partial Differential Equations: Analysis and Computations ('22)
- Bella, Giunti, Otto. *Effective multipoles in random media*, Communications in Partial Differential Equations ('20)

Thank you for your attention