Theoretical and Numerical Aspects of Anisotropic Singular Perturbations For Elliptic Problems CANUM 2024

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Abstract

We study singular perturbations of some class of linear elliptic problems when the diffusion coefficient is very small in some directions . We prove global estimates on the rate of convergence of the solution toward its limit, and we show uniform estimates for Q_1 finite element scheme.

Introduction

Let $\Omega = \omega_1 \times \omega_2$ be a bounded Lipschitz domain of $\mathbb{R}^N$, where ω_1 and ω_2 are two Lipschitz domains of $\mathbb{R}^q$ and $\mathbb{R}^{N-q}$ respectively, with $N > q \geq 1$. A general linear elliptic problem is given by :

$$\begin{cases} u_\epsilon \in H_0^1(\Omega) \\ \int_\Omega A_\epsilon \nabla u_\epsilon \cdot \nabla \varphi dx = \int_\Omega f \varphi dx, \forall \varphi \in H_0^1(\Omega). \end{cases} \tag{1}$$

The diffusion matrix A_ϵ, $\epsilon \in (0,1]$ is given by

$$A_\epsilon = \begin{pmatrix} \epsilon^2 A_{11} & \epsilon A_{12} \\ \epsilon A_{21} & A_{22} \end{pmatrix},$$

where A_{11}, A_{22} are $q \times q$ and $(N-q) \times (N-q)$ matrices respectively. The limit problem of (1) is given by: For a.e. $X_1 \in \omega_1$,

$$\begin{cases} u(X_1, \cdot) \in H_0^1(\omega_2) \\ \int_{\omega_2} A_{22}(X_1, \cdot) \nabla_{X_2} u(X_1, \cdot) \cdot \nabla_{X_2} \varphi dX_2 = \int_{\omega_2} f(X_1, \cdot) \varphi dX_2, \forall \varphi \in H_0^1(\omega_2). \end{cases} \tag{2}$$

We are interested in the asymptotic behaviour of u_ϵ in the limit $\epsilon \to 0$. When $f \in L^2(\Omega)$, the problem (1) has been studied in [7]. For $f \in L^p(\Omega)$ with $1 < p < 2$, see [5]. For more regularity of the asymptotic behaviour, see [3] and [5] . For some nonlinear problems, see [4] and [5]. For the asymptotic behaviour of the semigroup generated, see [2]. **We suppose that:**

- There exists $\lambda > 0$ such that for a.e. $x \in \Omega$:
$$\forall \xi \in \mathbb{R}^N : A_1(x)\xi \cdot \xi \geq \lambda |\xi|^2. \tag{3}$$
- A_1 is bounded:
$$\forall (i,j) \in \{1,2,....,N\}^2 : a_{ij} \in L^\infty(\Omega). \tag{4}$$
- f is square integrable
$$f \in L^2(\Omega). \tag{5}$$
- f is regular in the X_1 direction.
$$\nabla_{X_1} f \in L^2(\Omega)^q. \tag{6}$$
- The block A_{12} is regular in the following sense
$$\partial_{x_i} a_{ij} \in L^\infty(\Omega), \partial_{x_j} a_{ij} \in L^\infty(\Omega) \text{ for } i = 1, ..., q \text{ and } j = q+1, ..., N. \tag{7}$$

In [7] and under the hypothesis (3), (4), (5), (6), (7), and $\nabla_{X_1} A_{22} \in L^\infty(\Omega)^{q(N-q)^2}$, the authors proved that for every $\omega_1' \subset\subset \omega_1$ open: :

$$\left\| \nabla_{X_2}(u_\epsilon - u) \right\|_{L^2(\omega_1' \times \omega_2)} = O(\epsilon) \text{ and } \left\| \nabla_{X_1}(u_\epsilon - u) \right\|_{L^2(\omega_1' \times \omega_2)} = O(1).$$

Our contribution consists in extending these estimates to the whole domain Ω, which will allow us to study the problem numerically. To obtain such results we will suppose some additional assumptions:

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$$\text{The block } A_{22} \text{ depends only on } X_2. \tag{8}$$
- We suppose that A_1 satisfies the regularity assumption
$$a_{ij} \in W^{1,\infty}(\bar{\Omega}) \text{ for any } (i,j) \in \{1,2,....,N\}^2, \tag{9}$$

  and the boundary condition
$$\text{For every } i \neq j : a_{ij} = 0 \text{ on } \partial\Omega. \tag{10}$$

1 Theoretical results

We introduce the Hilbert spaces:

$$H_0^1(\Omega;\omega_2) = \left\{ v \in L^2(\Omega) \text{ such that } \nabla_{X_2} v \in L^2(\Omega)^{N-q} \text{ and for a.e. } X_1 \in \omega_1,\ v(X_1, \cdot) \in H_0^1(\omega_2) \right\},$$

$$H_0^1(\Omega;\omega_1) = \left\{ v \in L^2(\Omega) \text{ such that } \nabla_{X_1} v \in L^2(\Omega)^{q} \text{ and for a.e. } X_2 \in \omega_2,\ v(\cdot, X_2) \in H_0^1(\omega_1) \right\},$$

normed by $\left\|\nabla_{X_2}(\cdot)\right\|_{L^2(\Omega)^{N-q}}$ and $\left\|\nabla_{X_1}(\cdot)\right\|_{L^2(\Omega)^q}$ respectively.

Theorem 1. [2], [1] *Suppose that the assumptions* (3), (4), (7), *and* (8) *hold then:*
1) For $f \in H_0^1(\Omega, \omega_1)$, there exists $C_{\lambda,\Omega,A_1} > 0$ such that:

$$\left\| \nabla_{X_2}(u_\epsilon - u) \right\|_{L^2(\Omega)^{N-q}} \leq C_{\lambda,\Omega,A_1} (\left\| \nabla_{X_1} f \right\|_{L^2(\Omega)^q} + \|f\|_{L^2(\Omega)}) \times \epsilon, \tag{11}$$

where u_ϵ is the unique solution of (1) *in $H_0^1(\Omega)$ and u is the unique solution to* (2) *in $H_0^1(\Omega;\omega_2)$, moreover we have $u \in H_0^1(\Omega)$.*
2) For $f \in L^r(\Omega)$ for some $\infty > r > 2$ such that (6) *is satisfied, there exists $C_{\lambda,f,A_1,\Omega} > 0$ such that*

$$\left\| \nabla_{X_2}(u_\epsilon - u) \right\|_{L^2(\Omega)^{N-q}} \leq C_{\lambda,f,A_1,\Omega} \times \epsilon^{\frac{1}{2} - \frac{1}{r}}. \tag{12}$$

In particular, when $r = \infty$ we have

$$\left\| \nabla_{X_2}(u_\epsilon - u) \right\|_{L^2(\Omega)^{N-q}} \leq C_{\lambda,f,A_1,\Omega} \times \epsilon^{\frac{1}{2}}.$$

Idea of the proof

1) The key of the proof is based on the use of tensor products. First, we prove (11) when $f \in H_0^1(\omega_1) \otimes H_0^1(\omega_2)$, and we conclude by the density of $H_0^1(\omega_1) \otimes H_0^1(\omega_2)$ in $H_0^1(\Omega, \omega_1)$.

2) We decompose f as $f = f_{1,\delta} + f_{2,\delta}$ such that $f_{1,\delta} \in H_0^1(\Omega, \omega_1)$, and $f_{2,\delta} \in L^2(\Omega))$, with $\|\nabla_{X_1} f_{1,\delta}\|_{L^2(\Omega)} \leq c_1 \delta^{-\frac{1}{2} - \frac{1}{r}}$ and $\|f_{2,\delta}\|_{L^2(\Omega)} \leq c_2 \delta^{\frac{1}{2} - \frac{1}{r}}$ and we use (11) and the linearity of the problem, then we take $\delta = \epsilon$.

2 Numerical Results

We assume that $N \in \{2,3\}$ and that the computational domain is $\Omega = (0,1)^N$. We define a rectangular mesh $(\mathcal{R}_h)$, $0 < h \leq 1$ of Ω We denote $\mathbb{Q}_1(R)$ the space of real polynomials in N variables of partial degree less or equal to 1 over $R \subset \mathbb{R}^N$

$$W_h = \left\{ v \in C^0(\bar{\Omega}),\ v_{|R} \in \mathbb{Q}_1(R) \text{ for any } R \in \mathcal{R}_h \right\},$$

and

$$V_h = \{ v \in W_h, \text{ and } v = 0 \text{ on } \partial\Omega \}.$$

The numerical schemes to approximate (1) and (2) are

$$\begin{cases} \int_\Omega A_\epsilon \nabla u_{\epsilon,h} \cdot \nabla v dx = \int_\Omega I_h(f) v dx, \quad \forall v \in V_h, \\ u_{\epsilon,h} \in V_h. \end{cases}, \tag{13}$$

and

$$\begin{cases} \int_\Omega A_{22} \nabla_{X_2} u_h \cdot \nabla_{X_2} v dx = \int_\Omega I_h(f) v dx, \quad \forall v \in V_h, \\ u_h \in V_h, \end{cases},$$

where $I_h : H^2(\Omega) \longrightarrow W_h$ is the classical interpolation operator.
The classical Céa's lemma gives an estimate of the form

$$\left\| \nabla_{X_2}(u_{\epsilon,h} - u_\epsilon) \right\|_{L^2(\Omega)^{N-q}} = C_{\lambda,f,\Omega,A_1} \frac{h}{\epsilon^4}. \tag{14}$$

To ensure a good numerical approximation of the exact solution u_ϵ when ϵ is very small, then one must take h much smaller than ϵ, which is impractical from the numerical point of view. To fix this problem, we must prove some uniform estimates which hold for some kind of numerical scheme called "asymptotically preserved".

Theorem 2. [1] *Let $\Omega = (0,1)^N$, with $N \in \{2,3\}$. Assume that A_1 satisfies (3), (8), (9), and (10).*
1) Let $f \in H^2(\Omega)$, then there exists a positive constant C_{λ,f,Ω,A_1} independent of h and ϵ such that

$$\|\nabla_{X_2}(u_{\epsilon,h} - u_\epsilon)\|_{L^2(\Omega)^{N-q}} \leq C_{\lambda,f,\Omega,A_1} h^{\frac{1}{5}}, \tag{15}$$

where $u_{\epsilon,h}$ and u_ϵ are the solutions of (13) and (1) respectively.
2) If we assume, in addition, that $f \in H_0^1(\Omega)$ then we have

$$\|\nabla_{X_2}(u_{\epsilon,h} - u_\epsilon)\|_{L^2(\Omega)^{N-q}} \leq C_{\lambda,f,\Omega,A_1} h^{\frac{1}{3}}. \tag{16}$$

Idea of the proof

1) We apply a simple method given in [6]. We combine estimates of the form

$$\|u_{\epsilon,h} - u_\epsilon\|_\Omega \leq C\frac{h}{\epsilon^\alpha}, \text{ and } \|u_{\epsilon,h} - u_\epsilon\|_\Omega \leq C(\epsilon^\beta + h^\gamma), \tag{17}$$

to obtain the $\epsilon-$uniform estimate

$$\|u_{\epsilon,h} - u_\epsilon\|_\Omega \leq C h^{\min(\frac{\beta}{\alpha+\beta}, \gamma)}.$$

2) For the first estimate in (17), we use a Céa's type lemma and the following regularity estimate [3]

$$\|\nabla^2_{X_2} u_\epsilon\|_{L^2(\Omega)^{(N-q)^2}} + \epsilon^2 \|\nabla^2_{X_1} u_\epsilon\|_{L^2(\Omega)^{q^2}} + \epsilon \|\nabla^2_{X_1 X_2} u_\epsilon\|_{L^2(\Omega)^{q(N-q)}} \leq C_{\lambda,\Omega,f,A_1}. \tag{18}$$

to obtain

$$\left\| \nabla_{X_2}(u_{\epsilon,h} - u_\epsilon) \right\|_{L^2(\Omega)^{N-q}} = C_{\lambda,\Omega,f,A_1} \frac{h}{\epsilon^2}. \tag{19}$$

3) We use the tensor product technique, and an appropriate decomposition of f to obtain the second estimate in (17), with $(\beta, \gamma) \in \{(1,1), (\frac{1}{2}, \frac{1}{4})\}$ depending on the degree of regularity of f.

	$\epsilon = 1$	$\epsilon = 0.75$	$\epsilon = 0.5$	$\epsilon = 0.1$	$\epsilon = 0.01$	$\epsilon = 10^{-6}$
$h = 0.1$	0.007211	0.009230	0.011537	0.014279	0.014420	0.014422
$h = 0.02$	0.001443	0.001847	0.002309	0.002858	0.002886	0.002886
$h = 0.01$	0.000721	0.000923	0.001154	0.001429	0.001443	0.001443
$h = 0.001$	7.21×10^{-5}	9.23×10^{-5}	0.000115	0.000142	0.000144	0.000144

Table 1: Numerical error results.

3 Forthcoming Research

- Extending (11) and (12) to general asymptotic expansion of order $m \geq 1$.
- Numerical study of the parabolic problem [2].
- Convergence for the linear problem with L^1 data.

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