The 2D discrete random matching problem

Nicolas CLOZEAU

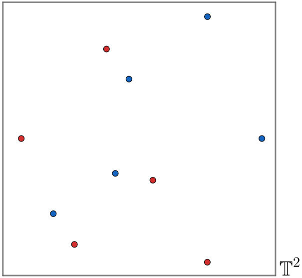
Institute of Science and Technology Austria

Joint work with Francesco MATTESINI

CANUM 2024

May 27-31, 2024

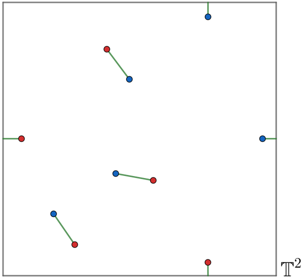
Quadratic 2D discrete random matching



Setting:

$$\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n \subset \mathbb{T}^2 \text{ random with } X_i, Y_i \sim \rho\, dx$$

Quadratic 2D discrete random matching



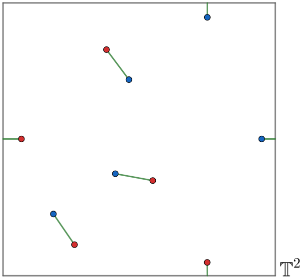
Setting:

$$\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n \subset \mathbb{T}^2 \text{ random with } X_i, Y_i \sim \rho\, \mathrm{d}x$$

Matching problem: Combinatorics optimization problem

$$\min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^{n} |X_i - Y_{\sigma(i)}|^2$$

Quadratic 2D discrete random matching



Setting:

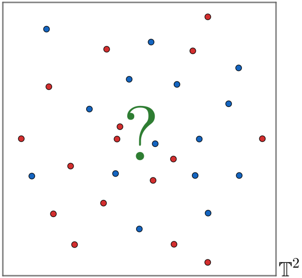
$$\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n \subset \mathbb{T}^2 \text{ random with } X_i, Y_i \sim \rho\, dx$$

Matching problem: Combinatorics optimization problem

$$\min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n |X_i - Y_{\sigma(i)}|^2$$

Problem appears in: statistical physics, computer science, economics, ...

Quadratic 2D discrete random matching



Setting:

$$\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n \subset \mathbb{T}^2 \text{ random with } X_i, Y_i \sim \rho\,\mathrm{d}x$$

Matching problem: Combinatorics optimization problem

$$\min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n |X_i - Y_{\sigma(i)}|^2$$

Problem appears in: statistical physics, computer science, economics, ...

Solution for large data $n \gg 1$: Combinatoric algorithms with polynomial complexity, Kuhn, Munkres (50′s)

A PDE approach to the matching problem

$$C_n := \min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^{n} |X_i - Y_{\sigma(i)}|^2$$

Optimal transport point of view: Caracciolo, Lucibello, Parisi and Sicuro (14')

$$C_n = n\, W_2^2\Big(\underbrace{\frac{1}{n}\sum_{k=1}^{n} \delta_{X_k}}_{:=\mu^n}, \underbrace{\frac{1}{n}\sum_{k=1}^{n} \delta_{Y_k}}_{:=\nu^n} \Big)$$

$$W_2^2(\mu^n,\nu^n) = \inf_{\pi} \left\{ \int_{\mathbb{T}^2\times\mathbb{T}^2} |x-y|^2 \, \mathrm{d}\pi(x,y) \,\middle|\, \pi_x = \mu^n \text{ and } \pi_y = \nu^n \right\}$$

A PDE approach to the matching problem

$$C_n := \min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^{n} |X_i - Y_{\sigma(i)}|^2$$

Optimal transport point of view: Caracciolo, Lucibello, Parisi and Sicuro (14')

$$C_n = n\, W_2^2\Bigg(\underbrace{\frac{1}{n}\sum_{k=1}^{n} \delta_{X_k}}_{:=\mu^n}, \underbrace{\frac{1}{n}\sum_{k=1}^{n} \delta_{Y_k}}_{:=\nu^n} \Bigg)$$

$$W_2^2(\mu^n, \nu^n) = \inf_{\pi} \left\{ \int_{\mathbb{T}^2 \times \mathbb{T}^2} |x-y|^2 \, \mathrm{d}\pi(x,y) \,\middle|\, \pi_x = \mu^n \text{ and } \pi_y = \nu^n \right\}$$

Question: Compute the optimal coupling π^n for $n \gg 1$

A PDE approach to the matching problem

$$W_2^2\Big(\underbrace{\frac{1}{n}\sum_{k=1}^{n}\delta_{X_k}}_{:=\mu^n},\underbrace{\frac{1}{n}\sum_{k=1}^{n}\delta_{Y_k}}_{:=\nu^n}\Big)$$

We solve the optimal transport problem: Assume $\mu^n = \mu^n\,\mathrm{d}x$ and $\nu^n = \nu^n\,\mathrm{d}x$

$$\pi^n = (\mathrm{Id}, T^n)\#\mu^n \quad \text{(Benamou-Brenier)}$$

and

$$T^n = \mathrm{Id} + \nabla h^n \quad \text{with} \quad \nu^n = (\mu^n \circ T^n)\det(\mathrm{Id} + \nabla^2 h^n) \quad \text{(Monge-Ampère)}$$

A PDE approach to the matching problem

$$W_2^2\Big(\underbrace{\frac{1}{n}\sum_{k=1}^{n}\delta_{X_k}}_{:=\mu^n},\underbrace{\frac{1}{n}\sum_{k=1}^{n}\delta_{Y_k}}_{:=\nu^n}\Big)$$

We solve the optimal transport problem: Assume $\mu^n = \mu^n\,\mathrm{d}x$ and $\nu^n = \nu^n\,\mathrm{d}x$

$$\pi^n = (\mathrm{Id}, T^n)\#\mu^n \quad \text{(Benamou-Brenier)}$$

and

$$T^n = \mathrm{Id} + \nabla h^n \quad \text{with} \quad \nu^n = (\mu^n \circ T^n)\det(\mathrm{Id} + \nabla^2 h^n) \quad \text{(Monge-Ampère)}$$

Approximation as $n \uparrow \infty$: Under ergodicity (say i.i.d.)

$$\mu^n, \nu^n \rightharpoonup \rho \quad \Longrightarrow \quad |(\nabla h^n, \nabla^2 h^n)| \ll 1 \text{ as } n \uparrow \infty$$

A PDE approach to the matching problem

$$W_2^2\Big(\underbrace{\frac{1}{n}\sum_{k=1}^{n}\delta_{X_k}}_{:=\mu^n},\underbrace{\frac{1}{n}\sum_{k=1}^{n}\delta_{Y_k}}_{:=\nu^n}\Big)$$

We solve the optimal transport problem: Assume $\mu^n = \mu^n\,\mathrm{d}x$ and $\nu^n = \nu^n\,\mathrm{d}x$

$$\pi^n = (\mathrm{Id}, T^n)\#\mu^n \quad \text{(Benamou-Brenier)}$$

and

$$T^n = \mathrm{Id} + \nabla h^n \quad \text{with} \quad \nu^n = (\mu^n \circ T^n)\det(\mathrm{Id} + \nabla^2 h^n) \quad \text{(Monge-Ampère)}$$

Approximation as $n \uparrow \infty$: Under ergodicity (say i.i.d.)

$$\mu^n, \nu^n \rightharpoonup \rho \quad \Longrightarrow \quad |(\nabla h^n, \nabla^2 h^n)| \ll 1 \text{ as } n \uparrow \infty$$

We expand

$$\mu^n \circ T^n \approx \mu^n + \nabla\rho\cdot\nabla h^n \quad \text{and} \quad \det(\mathrm{Id} + \nabla^2 h^n) \approx 1 + \Delta h^n$$

A PDE approach to the matching problem

$$W_2^2\Big(\underbrace{\frac{1}{n}\sum_{k=1}^{n}\delta_{X_k}}_{:=\mu^n},\underbrace{\frac{1}{n}\sum_{k=1}^{n}\delta_{Y_k}}_{:=\nu^n}\Big)$$

We solve the optimal transport problem: Assume $\mu^n = \mu^n \, dx$ and $\nu^n = \nu^n \, dx$

$$\pi^n = (\mathrm{Id}, T^n)\#\mu^n \quad \text{(Benamou-Brenier)}$$

and

$$T^n = \mathrm{Id} + \nabla h^n \quad \text{with} \quad \nu^n = (\mu^n \circ T^n)\det(\mathrm{Id} + \nabla^2 h^n) \quad \text{(Monge-Ampère)}$$

Approximation as $n \uparrow \infty$: Under ergodicity (say i.i.d.)

$$\mu^n, \nu^n \rightharpoonup \rho \quad \Longrightarrow \quad |(\nabla h^n, \nabla^2 h^n)| \ll 1 \text{ as } n \uparrow \infty$$

We expand

$$\mu^n \circ T^n \approx \mu^n + \nabla\rho \cdot \nabla h^n \quad \text{and} \quad \det(\mathrm{Id} + \nabla^2 h^n) \approx 1 + \Delta h^n$$

Plugging in Monge-Ampère

$$\nu^n = \mu^n + \nabla \cdot \rho \nabla h^n + \text{higher order terms}$$

A PDE approach to the matching problem

$$W_2^2\Big(\underbrace{\frac{1}{n}\sum_{k=1}^{n}\delta_{X_k}}_{:=\mu^n},\underbrace{\frac{1}{n}\sum_{k=1}^{n}\delta_{Y_k}}_{:=\nu^n}\Big)$$

We solve the optimal transport problem: Assume $\mu^n = \mu^n\,\mathrm{d}x$ and $\nu^n = \nu^n\,\mathrm{d}x$

$$\pi^n = (\mathrm{Id}, T^n)\#\mu^n \quad \text{(Benamou-Brenier)}$$

and

$$T^n = \mathrm{Id} + \nabla h^n \quad \text{with} \quad \nu^n = (\mu^n \circ T^n)\det(\mathrm{Id} + \nabla^2 h^n) \quad \text{(Monge-Ampère)}$$

Approximation as $n \uparrow \infty$: Under ergodicity (say i.i.d.)

$$\mu^n, \nu^n \rightharpoonup \rho \quad \Longrightarrow \quad |(\nabla h^n, \nabla^2 h^n)| \ll 1 \text{ as } n \uparrow \infty$$

We expand

$$\mu^n \circ T^n \approx \mu^n + \nabla\rho\cdot\nabla h^n \quad \text{and} \quad \det(\mathrm{Id} + \nabla^2 h^n) \approx 1 + \Delta h^n$$

Plugging in Monge-Ampère

$$\boxed{\nu^n = \mu^n + \nabla\cdot\rho\nabla h^n} + \text{~~higher order terms~~}$$

A PDE approach to the matching problem

$$C_n = n \int_{\mathbb{T}^2 \times \mathbb{T}^2} |x-y|^2 \mathrm{d}\pi^n(x,y)$$

To sum up:

$$\pi^n = (\mathrm{Id}, T^n)\#\mu^n \quad \text{with } T^n \approx \mathrm{Id} + \nabla h^n \text{ and } -\nabla \cdot \rho \nabla h^n = \mu^n - \nu^n$$

A PDE approach to the matching problem

$$C_n = n \int_{\mathbb{T}^2 \times \mathbb{T}^2} |x-y|^2 \mathrm{d}\pi^n(x,y)$$

To sum up:

$$\pi^n = (\mathrm{Id}, T^n)\#\mu^n \quad \text{with } T^n \approx \mathrm{Id} + \nabla h^n \text{ and } -\nabla \cdot \rho \nabla h^n = \mu^n - \nu^n$$

Successfully applied: Ambrosio, Stra, Trevisan, Goldman ('19, '22) to study the cost for $\rho = 1$

$$\mathbb{E}[C_n] \approx n\,\mathbb{E}\left[\int_{\mathbb{T}^2} |\nabla h^n|^2\right] \approx \frac{1}{2\pi}\log(n)$$

Generalise previous two sided estimates by Ajtai, Komlós and Tusnády ('84)

A PDE approach to the matching problem

$$C_n = n \int_{\mathbb{T}^2 \times \mathbb{T}^2} |x - y|^2 \mathrm{d}\pi^n(x, y)$$

To sum up:

$$\pi^n = (\mathrm{Id}, T^n)\#\mu^n \quad \text{with } T^n \approx \mathrm{Id} + \nabla h^n \text{ and } -\nabla \cdot \rho \nabla h^n = \mu^n - \nu^n$$

Successfully applied: Ambrosio, Stra, Trevisan, Goldman ('19, '22) to study the cost for $\rho = 1$

$$\mathbb{E}[C_n] \approx n\,\mathbb{E}\left[\int_{\mathbb{T}^2} |\nabla h^n|^2 \right] \approx \frac{1}{2\pi} \log(n)$$

Generalise previous two sided estimates by Ajtai, Komlós and Tusnády ('84)

Question: Can we justify the approximation of π^n, i.e.

$$\lim_{n\uparrow\infty} W_2^2\left(\pi^n, (\mathrm{Id}, \mathrm{Id} + \nabla h^n)\#\mu^n\right) = 0 \quad ?$$

A rigorous approach to the PDE ansatz

Consider $\{\mathrm{P}_t\}_{t>0}$ the heat semi-group and

$$-\nabla \cdot \rho \nabla h^{n,t} = \mu^{n,t} - \nu^{n,t} \quad \text{with} \int_{\mathbb{T}^2} h^{n,t} = 0$$

$$\mu^{n,t} := \mathrm{P}_t \mu^n \quad \text{and} \quad \nu^{n,t} := \mathrm{P}_t \nu^n$$

A rigorous approach to the PDE ansatz

Consider $\{\mathrm{P}_t\}_{t>0}$ the heat semi-group and

$$-\nabla \cdot \rho \nabla h^{n,t} = \mu^{n,t} - \nu^{n,t} \quad \text{with} \int_{\mathbb{T}^2} h^{n,t} = 0$$

$$\mu^{n,t} := \mathrm{P}_t \mu^n \quad \text{and} \quad \nu^{n,t} := \mathrm{P}_t \nu^n$$

Theorem (C., Mattesini, PTRF ('24))

Assume that $\rho \in \mathrm{H}^\varepsilon$ *with* $\varepsilon > 0$, $0 < \lambda \le \rho \le \Lambda$ *and for* $c > 0$, $\eta \in (0, \infty]$

$$\sup_{|F| \le 1} \left| \int_{\mathbb{T}^2 \times \mathbb{T}^2} F \, \mathrm{d}(\mathbb{P}_{(X_i, X_j)} - \mathbb{P}_{X_i} \otimes \mathbb{P}_{X_j}) \right| \lesssim \exp(-c|i-j|^\eta)$$

likewise for $\{Y_i\}_i$. *For* $t = \frac{\log^\gamma(n)}{n}$ *and* $\gamma \gg 1$, *it holds*

$$\mathbb{E}\left[W_2^2\left(\pi^n, \left(\mathrm{Id}, \mathrm{Id} + \nabla h^{n,t}\right) \# \mu^{n,t}\right)\right] \lesssim \frac{\log(n)}{n} \sqrt{\frac{\log\log(n)}{\log(n)}}$$

Generalise previous estimates by Ambrosio, Glaudo and Trevisan ('19) for $\eta = \infty$ and $\rho = 1$

A proof combining tools from optimal transport and PDEs

$$-\nabla \cdot \rho \nabla h^{n,t} = \mu^{n,t} - \nu^{n,t}, \quad T^n \# \mu^n = \nu^n \quad \text{and} \quad \hat{T}^{n,t} := \mathrm{Id} + \nabla h^{n,t}$$

Stability of transport maps: Compare

$$\pi^n = (\mathrm{Id}, T^n)\#\mu^n \quad \text{and} \quad \hat{\pi}^{n,t} := (\mathrm{Id}, \hat{T}^{n,t})\#\mu^{n,t}$$

A proof combining tools from optimal transport and PDEs

$$-\nabla \cdot \rho \nabla h^{n,t} = \mu^{n,t} - \nu^{n,t}, \quad T^n \# \mu^n = \nu^n \quad \text{and} \quad \hat{T}^{n,t} := \mathrm{Id} + \nabla h^{n,t}$$

Stability of transport maps: Compare

$$\pi^n = (\mathrm{Id}, T^n)\#\mu^n \quad \text{and} \quad \hat{\pi}^{n,t} := (\mathrm{Id}, \hat{T}^{n,t})\#\mu^{n,t}$$

If

$$\mu^n - \mu^{n,t} \underset{n\uparrow\infty}{\rightharpoonup} 0 \quad \text{and} \quad \nu^n - \hat{T}^{n,t}\#\mu^{n,t} \underset{n\uparrow\infty}{\rightharpoonup} 0$$

then

$$\pi^n - \hat{\pi}^{n,t} \underset{n\uparrow\infty}{\rightharpoonup} 0$$

A proof combining tools from optimal transport and PDEs

$$-\nabla \cdot \rho \nabla h^{n,t} = \mu^{n,t} - \nu^{n,t}, \quad T^n \# \mu^n = \nu^n \quad \text{and} \quad \hat{T}^{n,t} := \mathrm{Id} + \nabla h^{n,t}$$

Stability of transport maps: Compare

$$\pi^n = (\mathrm{Id}, T^n)\#\mu^n \quad \text{and} \quad \hat{\pi}^{n,t} := (\mathrm{Id}, \hat{T}^{n,t})\#\mu^{n,t}$$

If

$$\mu^n - \mu^{n,t} \underset{n\uparrow\infty}{\rightharpoonup} 0 \quad \text{and} \quad \nu^n - \hat{T}^{n,t}\#\mu^{n,t} \underset{n\uparrow\infty}{\rightharpoonup} 0$$

then

$$\pi^n - \hat{\pi}^{n,t} \underset{n\uparrow\infty}{\rightharpoonup} 0$$

Quantitative version: Ambrosio, Glaudo and Trevisan ('19): There exists $c > 0$ such that provided

$$|(\nabla h^{n,t}, \nabla^2 h^{n,t})| \le c$$

it holds

$$W_2^2(\pi^n, \hat{\pi}^{n,t}) \lesssim W_2^2(\mu^{n,t}, \mu^n) + W_2^2(\nu^{n,t}, \nu^n) + W_2^2(\nu^{n,t}, \hat{T}^{n,t}\#\mu^{n,t})$$

A proof combining tools from optimal transport and PDEs

$$-\nabla \cdot \rho \nabla h^{n,t} = \mu^{n,t} - \nu^{n,t}, \quad T^n \# \mu^n = \nu^n \quad \text{and} \quad \hat{T}^{n,t} := \mathrm{Id} + \nabla h^{n,t}$$

Regularity estimates: For the choice $t = \frac{\log^\gamma(n)}{n}$ for $\gamma \gg 1$

$$\mathbb{P}\left(\|(\nabla h^{n,t}, \nabla^2 h^{n,t})\|_{\mathrm{L}^\infty} \leq \frac{1}{\log(n)} \right) \geq 1 - o\left(\frac{1}{n^\ell}\right) \quad \text{for any } \ell \geq 1$$

A proof combining tools from optimal transport and PDEs

$$-\nabla \cdot \rho \nabla h^{n,t} = \mu^{n,t} - \nu^{n,t}, \quad T^n \# \mu^n = \nu^n \quad \text{and} \quad \hat{T}^{n,t} := \mathrm{Id} + \nabla h^{n,t}$$

Regularity estimates: For the choice $t = \frac{\log^{\gamma}(n)}{n}$ for $\gamma \gg 1$

$$\mathbb{P}\left(\|(\nabla h^{n,t}, \nabla^2 h^{n,t})\|_{\mathrm{L}^\infty} \leq \frac{1}{\log(n)} \right) \geq 1 - o\left(\frac{1}{n^{\ell}}\right) \quad \text{for any } \ell \geq 1$$

Contractivity error $W_2^2(\mu^{n,t}, \mu^n)$**:**

$$\mathbb{E}\left[W_2^2(\mu^{n,t}, \mu^n) \right] \lesssim \frac{\log\log(n)}{n} + t\|\rho_t - \rho\|_{\mathrm{L}^1}$$

Improvement of the classical estimate $\lesssim t$

A proof combining tools from optimal transport and PDEs

$$-\nabla \cdot \rho \nabla h^{n,t} = \mu^{n,t} - \nu^{n,t}, \quad T^n \# \mu^n = \nu^n \quad \text{and} \quad \hat{T}^{n,t} := \mathrm{Id} + \nabla h^{n,t}$$

Moser coupling error $W_2^2(\nu^{n,t}, \hat{T}^{n,t} \# \mu^{n,t})$: By Benamou-Brenier' theorem

$$\nu^{n,t} = \phi(1, \cdot) \# \mu^{n,t} \quad \text{with } \phi \text{ flow induced by } s \mapsto \frac{\rho \nabla h^{n,t}}{s\mu^{n,t} + (1-s)\nu^{n,t}}$$

A proof combining tools from optimal transport and PDEs

$$-\nabla \cdot \rho \nabla h^{n,t} = \mu^{n,t} - \nu^{n,t}, \quad T^n \# \mu^n = \nu^n \quad \text{and} \quad \hat{T}^{n,t} := \mathrm{Id} + \nabla h^{n,t}$$

Moser coupling error $W_2^2(\nu^{n,t}, \hat{T}^{n,t} \# \mu^{n,t})$**:** By Benamou-Brenier' theorem

$$\nu^{n,t} = \phi(1, \cdot) \# \mu^{n,t} \quad \text{with } \phi \text{ flow induced by } s \mapsto \frac{\rho \nabla h^{n,t}}{s\mu^{n,t} + (1-s)\nu^{n,t}}$$

$$\begin{aligned} W_2^2(\nu^{n,t}, \hat{T}^{n,t} \# \mu^{n,t}) &= W_2^2(\phi(1, \cdot) \# \mu^{n,t}, \hat{T}^{n,t} \# \mu^{n,t}) \\ &\leq \int_{\mathbb{T}^2} |\phi(1, \cdot) - (\mathrm{Id} + \nabla h^{n,t})|^2 \\ &\lesssim \int_{\mathbb{T}^2} |\rho_t - \rho|^2 |\nabla h^{n,t}|^2 \end{aligned}$$

References

The paper: C., Mattesini. *Annealed quantitative estimates for the quadratic 2D-discrete random matching problem*, Probability theory and related fields ('24)

Related works:

- Caracciolo, Lucibello, Parisi, Sicuro. *Scaling hypothesis for the Euclidean bipartite matching problem*, Physical Review E ('14)
- Ambrosio, Stra, Trevisan. *A PDE approach to a 2-dimensional matching problem*, Probability theory and related fields ('19)
- Ambrosio, Glaudo, Trevisan. *On the optimal map in the 2-dimensional random matching problem*, Discrete & Continuous Dynamical Systems: Series A ('19)

Thank you for your attention !