A DDFV Scheme for Incompressible Two-Phase Flow Degenerate Problem in Porous Media

Thomas Crozon[1] with El-Houssaine Quenjel[2] and Mazen Saad[1].

[1]Ecole Centrale de Nantes,Jean Leray Mathematics Laboratory,
[2]Chair of Biotechnology, LGPM, CentraleSup elec, CEBB

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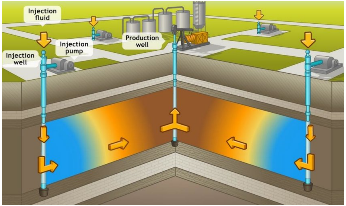
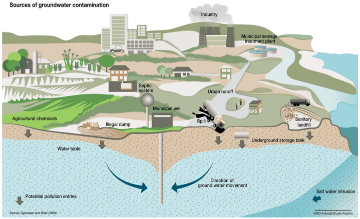
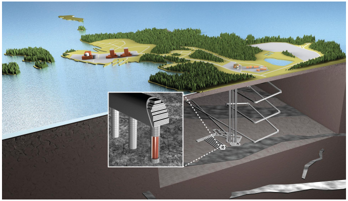
Many applications



Secondary and tertiary oil recovery [1]



Sources of groundwater contaminations [2]



Nuclear waster management [3]

[1] Savory, Luke. (2015). Enhanced oil recovery by flooding with aqueous surfactant solution : a model study and comparison with theory.
[2] https://www.grida.no/resources/13721
[3] https://www.wired.co.uk/article/finland-bury-nuclear-waste

Porous media

Ω a 2D bounded polygonal domain. $Q_{t_f} = \Omega \times (0, t_f)$.
Two phases $\{nw, w\}$

Microscopic/macroscopic view

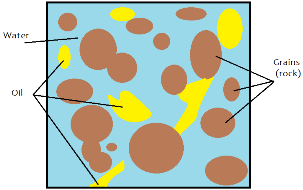
- Porosity $\quad \phi(x) = \dfrac{\text{fluids volume}}{\text{total volume}}$,
- Permeability tensor $\Lambda(x)$ symmetric positive-definite matrix.

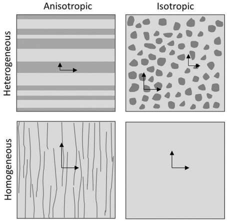
Unknowns s_{nw}, s_w, p_{nw}, p_w

- Saturations $s_\alpha = \dfrac{\alpha\text{-phase volume}}{\text{fluids volume}}$,
- Phases pressures p_α.

Saturing phases

$$\boxed{s_{nw} + s_w = 1.}$$





Isotropic/anisotropic [1]

[1] https://explorer.aapg.org/story/articleid/45535/the-fabric-or-internal-structure-of-rocks-part-2

Diphasic Darcy flow model in porous media

ϕ porosity,
s_α saturation,
ρ_α volumetric mass,
$\mathbf{V}_\alpha$ phase velocity,
q^α source term

Mass conservation equations

$$\begin{aligned} \phi\partial_t(\rho_w s_w) + \mathrm{div}(\rho_w \mathbf{V}_w) + \rho_w q^w &= 0 \quad \text{in} \quad Q_{t_f}, \\ \phi\partial_t(\rho_{nw} s_{nw}) + \mathrm{div}(\rho_{nw} \mathbf{V}_{nw}) + \rho_{nw} q^{nw} &= 0 \quad \text{in} \quad Q_{t_f}. \end{aligned}$$

Incompressible: ρ_α constant/compressible: $\rho_\alpha(p_\alpha)$.

$M_\alpha(s_\alpha)$ phase mobility,
Λ permeability tensor,
g gravitationnal acceleration

Diphasic Darcy law

$$\mathbf{V}_\alpha = -M_\alpha(s_\alpha)\Lambda(\nabla p_\alpha - \rho_\alpha \vec{g}).$$

Capillary pressure

$$P_c(s_{nw}) = p_{nw} - p_w.$$

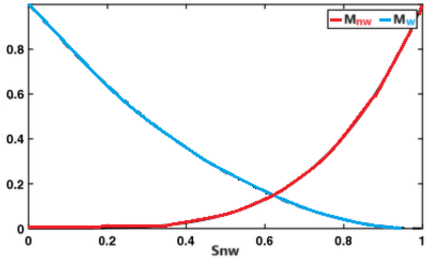
And initial and boundaries conditions (Dirichlet, Neumann, Outflow) ...

Degeneracy issue

Mobilities

- M_α continuous, increasing with s_α,
- continuously extended mobilities to $\mathbb{R}$ with the values at 0 and 1,
- the degeneracy issue:

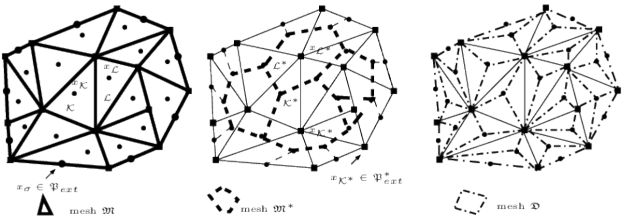
$$M_\alpha(s_\alpha = 0) = 0$$



$$\mathbf{V}_\alpha = -M_\alpha(s_\alpha)\Lambda(\nabla p_\alpha - \rho_\alpha \vec{g})$$

∇p_α can't be controlled when $s_\alpha = 0$!

DDFV framework



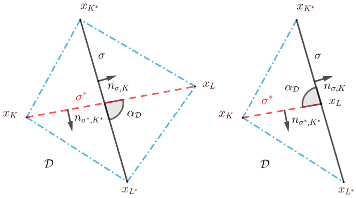
DDFV meshes building [1] Boyer, Franck Hubert, Florence. (2008)...

[Andreianov et al,2007, Domelevo and Omnes, 2005, Coudiere et al, 2008, Krell, 2010.]

F.V. functions $u_{\mathcal{T}} = ((u_K)_{K \in \overline{\mathfrak{M}}}, (u_{K^*})_{K^* \in \overline{\mathfrak{M}^*}}) \in \mathbb{R}^{\mathcal{T}}$.

DDFV gradient $\nabla^{\mathfrak{D}} u_{\mathcal{T}} = \sum_{\mathcal{D} \in \mathfrak{D}} \nabla^{\mathcal{D}} u_{\mathcal{T}} \mathbf{1}_{\mathcal{D}}$, with:

$$\nabla^{\mathcal{D}} u_{\mathcal{T}} = \frac{1}{\sin(\alpha_{\mathcal{D}})} \left(\frac{u_L - u_K}{m_{\sigma^*}} \vec{n}_{\sigma,K} + \frac{u_{L^*} - u_{K^*}}{m_\sigma} \vec{n}_{\sigma^*,K^*} \right).$$



Equivalently verifying the relations

$$\nabla^{\mathcal{D}} u_{\mathcal{T}} \cdot \vec{t}_{K,L} = \frac{u_L - u_K}{m_{\sigma^*}}, \quad \nabla^{\mathcal{D}} u_{\mathcal{T}} \cdot \vec{t}_{K^*,L^*} = \frac{u_{L^*} - u_{K^*}}{m_\sigma}.$$

Numerical scheme

Euler implicit finite volume scheme (integrating on $K \times [t^n, t^{n+1}[$

$$(1) \approx \int_{t^n}^{t^{n+1}} \int_K \phi(x) \frac{s_\alpha(x, t^{n+1}) - s_\alpha(x, t^n)}{\delta t} \,\mathrm{d}x\,\mathrm{d}t \approx \delta t \; m_K \phi_K \frac{s^{n+1}_{\alpha,K} - s^n_{\alpha,K}}{\delta t}.$$

$$(2) \approx \int_{t^n}^{t^{n+1}} \sum_{\sigma \in \mathcal{E}_K} \vec{V}_\alpha \cdot \vec{n}_{\sigma,K}(x, t^{n+1}) \,\mathrm{d}\sigma\,\mathrm{d}t \approx \delta t \sum_{\sigma = K|L \in \mathcal{E}_K} V^{\alpha,n+1}_{KL}.$$

Equations discretization A a cell, n a time-step:

$$\begin{aligned}
\phi_A \frac{m_A}{\delta t}(s^{n+1}_{nw,A} - s^n_{nw,A}) + \sum_{\sigma = A|L \in \mathcal{E}_A} V^{nw,n+1}_{AL} &= 0,\\
\phi_A \frac{m_A}{\delta t}(s^{n+1}_{w,A} - s^n_{w,A}) + \sum_{\sigma = A|L \in \mathcal{E}_A} V^{w,n+1}_{AL} &= 0,\\
P_c(s^{n+1}_{nw,A}) = p^{n+1}_{nw,A} - p^{n+1}_{w,A} \quad \text{and} \quad s^{n+1}_{nw,A} + s^{n+1}_{w,A} &= 1.
\end{aligned}$$

$+$ discrete boundary and initial conditions...

Velocity approximation

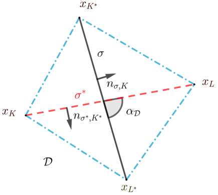
Velocity through the interface $\sigma = K|L$ (resp. $\sigma^* = K^*|L^*$):

$$V_{KL}^{\alpha,n+1} := \tau_{KL} M_{\alpha,KL}^{up,n+1}(p_{\alpha,L}^{n+1} - p_{\alpha,K}^{n+1}) + \eta_{\mathcal{D}} \sqrt{M_{\alpha,KL}^{min,n+1}} \sqrt{M_{\alpha,K^*L^*}^{up,n+1}}(p_{\alpha,L^*}^{n+1} - p_{\alpha,K^*}^{n+1}),$$

$$V_{K^*L^*}^{\alpha,n+1} := \tau_{K^*L^*} M_{\alpha,K^*L^*}^{up,n+1}(p_{\alpha,L^*}^{n+1} - p_{\alpha,K^*}^{n+1}) + \eta_{\mathcal{D}} \sqrt{M_{\alpha,K^*L^*}^{min,n+1}} \sqrt{M_{\alpha,KL}^{up,n+1}}(p_{\alpha,L}^{n+1} - p_{\alpha,K}^{n+1}).$$

With $$M_{\alpha,AB}^{up,n+1} := \begin{cases} M(s_{\alpha,B}^{n+1}) & \text{if } p_{\alpha,B}^{n+1} - p_{\alpha,A}^{n+1} \geq 0 \\ M_\alpha(s_{\alpha,A}^{n+1}) & \text{otherwise} \end{cases},$$

$$M_{\alpha,AB}^{min,n+1} := \min\left(M_\alpha(s_{\alpha,A}^{n+1}), M_\alpha(s_{\alpha,B}^{n+1})\right),$$



where $$\tau_{KL} = \frac{m_\sigma}{m_{\sigma^*}} \frac{\langle \Lambda_{\mathcal{D}} \vec{n}_{KL}, \vec{n}_{KL} \rangle}{\sin(\alpha_{\mathcal{D}})} > 0, \quad \tau_{K^*L^*} = \frac{m_{\sigma^*}}{m_\sigma} \frac{\langle \Lambda_{\mathcal{D}} \vec{n}_{K^*L^*}, \vec{n}_{K^*L^*} \rangle}{\sin(\alpha_{\mathcal{D}})} > 0,$$

$$\eta_{\mathcal{D}} = \frac{\langle \Lambda_{\mathcal{D}} \vec{n}_{KL}, \vec{n}_{K^*L^*} \rangle}{\sin(\alpha_{\mathcal{D}})}.$$

Maximum principle

Lemma Bound-preservation

Let $(p_{nw,\mathcal{T},\delta t}, p_{w,\mathcal{T},\delta t})$ a discrete solution. Then, for $\alpha \in \{nw, w\}$, the discrete saturation of the α-phase obeys its physical bounds i.e.

$$0 \leq s^{n+1}_{\alpha,K} \leq 1, \quad \forall K \in \mathcal{T}, \forall n \in 0, N-1.$$

Proof: $\alpha = nw$, $\quad K = argmin_{A \in \mathcal{T}} s^{n+1}_{nw,A}$, $\quad (s^{n+1}_{nw,K})^- = -\min(s^{n+1}_{nw,K}, 0) \geq 0.$

$$\phi_K \frac{m_K}{\delta t}\left(s^{n+1}_{nw,K} - s^{n}_{nw,K}\right)(s^{n+1}_{nw,K})^- - (s^{n+1}_{nw,K})^- \sum_{\sigma \in \mathcal{E}_K} M^{up,n+1}_{nw,KL} \tau_{KL}(p^{n+1}_{nw,L} - p^{n+1}_{nw,K})$$

$$+ \sqrt{M^{min,n+1}_{nw,KL}} \sqrt{M^{up,n+1}_{nw,K^*L^*}} \eta_{\mathcal{D}}(p^{n+1}_{nw,L^*} - p^{n+1}_{nw,K^*}) = 0.$$

- $\phi_K \left(s^{n+1}_{nw,K} - s^{n}_{nw,K}\right)(s^{n+1}_{nw,K})^- = -\phi_K \left(|(s^{n+1}_{nw,K})^-|^2 + s^{n}_{nw,K}(s^{n+1}_{nw,K})^-\right) \leq 0.$

Degeneracy implies $\quad M_{nw}(s^{n+1}_{nw})(s^{n+1}_{nw})^- = 0$

- $(s^{n+1}_{nw,K})^- \sqrt{M^{min,n+1}_{nw,KL}} \sqrt{M^{up,n+1}_{nw,K^*L^*}} \eta_{\mathcal{D}}(p^{n+1}_{nw,L^*} - p^{n+1}_{nw,K^*}) = 0,$
- Thanks to the upwinding $\quad (s^{n+1}_{nw,K})^- M^{up,n+1}_{nw,KL} \tau_{KL}(p^{n+1}_{nw,L} - p^{n+1}_{nw,K}) \geq 0.$

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Global pressure, capillary term

Introduce in [Chavent and Jaffré, 1986].
Total mobility $0 < m_0 \leq M(s_{nw}) = M_{nw}(s_{nw}) + M_w(1 - s_{nw})$.

Global pressure:

$$p = p_{nw} - \hat{p}_{nw}(s_{nw}) = p_w + \hat{p}_w(s_{nw}),$$

where the corrective pressures $\hat{p}_{nw}$, $\hat{p}_w$

$$\begin{cases} \hat{p}_{nw}(s_{nw}) = \displaystyle\int_0^{s_{nw}} \frac{M_w(1-u)}{M(u)} p_c'(u)\,\mathrm{d}u \\ \hat{p}_w(s_{nw}) = \displaystyle\int_0^{s_{nw}} \frac{M_{nw}(u)}{M(u)} p_c'(u)\,\mathrm{d}u \end{cases}.$$

Capillary term:

$$\xi(s_{nw}) = \int_0^{s_{nw}} \frac{\sqrt{M_w(1-u)M_{nw}(u)}}{M(u)} p_c'(u)\,\mathrm{d}u.$$

It verifies the continuous relation

$$\boxed{M_{nw}(s_{nw})|\nabla p_{nw}|^2 + M_w(s_w)|\nabla p_w|^2 = M(s_{nw})|\nabla p|^2 + M(s_{nw})|\nabla \xi(s_{nw})|^2.}$$

But, we don't keep it in the discrete world...

Lemma

For every A, B in $\mathcal{T}$, there holds:

$$m_0\left((p_B - p_A)^2 + (\xi_B - \xi_A)^2\right) \leq M_{nw,AB}^{up}\left(p_{nw,B} - p_{nw,A}\right)^2 + M_{w,AB}^{up}\left(p_{w,B} - p_{w,A}\right)^2.$$

Energy estimates

Proposition (Energy estimates)

Let $(p_{\alpha,\mathcal{T},\delta t})_{\alpha\in\{nw,w\}}$ be solution of the BP-DDFV. Then, there exists a constant C independent of the discretization parameters namely h and δt, but depending on the regularity upper bound, such that

$$\sum_{n=0}^{N-1} \delta t \left(\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} \left\| \nabla^{\mathcal{D}} p_{\mathcal{T}}^{n+1} \right\|^2 + \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} \left\| \nabla^{\mathcal{D}} \xi_{\mathcal{T}}^{n+1} \right\|^2 \right) \leq C.$$

Proof: • Variationnal formulation.

- Accumulation term: classic.
- The $\sqrt{M_\alpha^{up}}$ is to force the coercivity

$$V_{KL}^{\alpha,n+1}(p_{\alpha,L}^{n+1} - p_{\alpha,K}^{n+1}) + V_{K^*L^*}^{\alpha,n+1}(p_{\alpha,L^*}^{n+1} - p_{\alpha,K^*}^{n+1})$$
$$\geq \gamma \left(\tau_{KL} M_{\alpha,KL}^{up,n+1}(p_{\alpha,L}^{n+1} - p_{\alpha,K}^{n+1})^2 + \tau_{K^*L^*} M_{\alpha,K^*L^*}^{up,n+1}(p_{\alpha,L^*}^{n+1} - p_{\alpha,K^*}^{n+1})^2 \right).$$

- Then we use the previous lemma.

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Penalisation term and convergence (1/2)

For $\gamma > 0$, $\epsilon \in]0,2[$, a cell K

$$\phi_K\left(s_{\alpha,K}^{n+1} - s_{\alpha,K}^{n}\right) - \frac{\delta t}{m_K}\sum_{\sigma=K|L\in\mathcal{E}_K} V_{KL}^{\alpha,n+1} + \gamma\frac{\delta t}{m_K}\mathcal{P}_K p_{\alpha,\mathcal{T}}^{n+1} = 0$$

where

$$\mathcal{P}_K p_{\alpha,\mathcal{T}} = \frac{1}{h_{\mathfrak{D}}{}^{\epsilon}}\sum_{K^*\in\overline{\mathfrak{M}^*}} m_{K\cap K^*} M_{\alpha,KK^*}^{up}\left(p_{\alpha,K} - p_{\alpha,K^*}\right).$$

- Keep maximum principle.

Penalisation term and convergence (2/2)

Refined energy estimates:

$$\left\|\nabla^{\mathfrak{D}} p_{\mathcal{T},\delta t}\right\|_2^2 + \left\|\nabla^{\mathfrak{D}} \xi_{\mathcal{T},\delta t}\right\|_2^2$$
$$+ \frac{\gamma}{h_{\mathfrak{D}}^{\epsilon}} \left(\left\|\xi_{\mathfrak{M},\delta t} - \xi_{\overline{\mathfrak{M}^*},\delta t}\right\|_{L^2(Q_{t_f})}^2 + \left\|p_{\mathfrak{M},\delta t} - p_{\overline{\mathfrak{M}^*},\delta t}\right\|_{L^2(Q_{t_f})}^2 \right) \leq C,$$

with $C > 0$ depending on mesh regularity, and problem data, independent of the mesh size, and δt.

- convergence of the approximate solution to a weak solution of the model (Compacity results).

The test

Compressible case. $\Omega = (0, 1m)^2$. $\phi = 0,206$.

Boundary conditions:

- Left-down corner, high pressure water injected, Dirichlet conditions: $s_g = 0$, $p_w = 4,6732 \times 10^5 \ Pa$.
- Right-up corner overflow, outflow conditions. The fluids can come out at atmospheric pressure $p_g = P_{atm} = 1,013 \times 10^5 \ Pa$.
- Impervious boundaries everywhere else, Neumann conditions. Null *gas* and *water* normal flows.

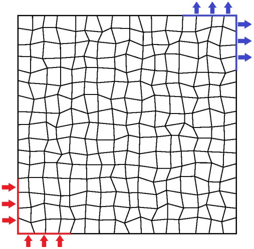
$P_c(s_g) = P_{max} s_g$ with $P_{max} = 10^5 \ Pa$.
$M_\alpha(s_\alpha) = s_\alpha^2/\mu_\alpha$ with $\mu_g = 9 \times 10^{-5} \ Pa\ s$, $\mu_w = 10^{-3} \ Pa\ s$.

Unknown p_w and s_g. We use a Newton method with preconditioning (BiCGStab).

w weakly compressible

$\rho_w(p_w) = \rho_{w,ref}\,(1 + c_{ref}(p_w - P_{w,ref}))$
$\rho_{w,ref} = 1000 \ kg\ m^{-3}$, $\quad c_{ref} = 10^{-6} Pa^{-1}$, $P_{w,ref} = 1.013 \times 10^5 \ Pa$.

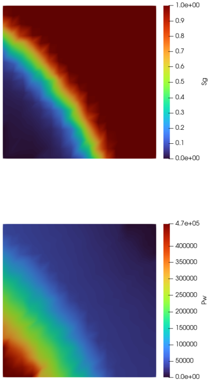
g perfect gas $\rho_g(p_g) = \rho_{g,ref} \frac{p_g}{P_{g,ref}}$ with $\rho_{g,ref} = 400 \ kg\ m^{-3}$, $P_{g,ref} = 1.013 \times 10^5 \ Pa$.



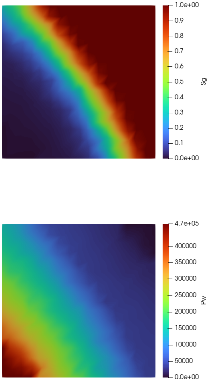
Quadrangle mesh with boundary conditions, 289 nodes, 256 primal cells.

Test 1: Uniform anistropic

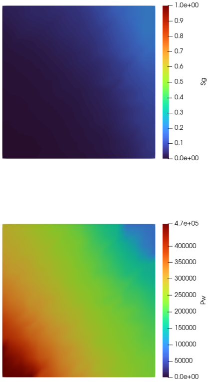
$$\Lambda = 0.15 \times 10^{-10} \times R_{\theta_0} \times \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \times R_{\theta_0}^{-1} \; [m^2]$$ with R_{θ_0} the rotation of angle $\theta_0 = \pi/6$. $t_f = 30\, s$, $\delta t = 0.005\, s$.



Gas saturation [-] (up) and water pressure [Pa] (down) at $t = 6\ s$

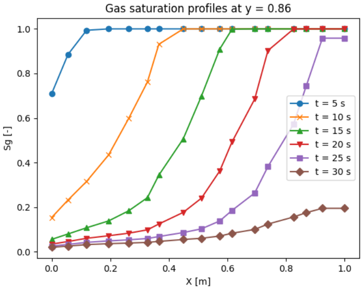


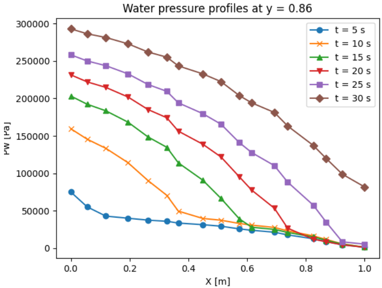
Gas saturation [-] (up) and water pressure [Pa] (down) at $t = 12\ s$

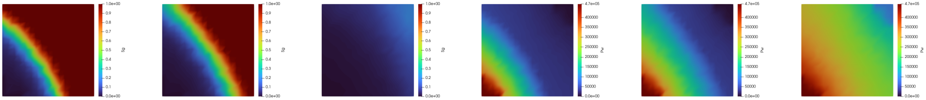


Gas saturation [-] (up) and water pressure [Pa] (down) at $t = 30\ s$

Test 1: profiles

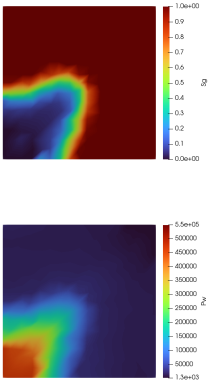




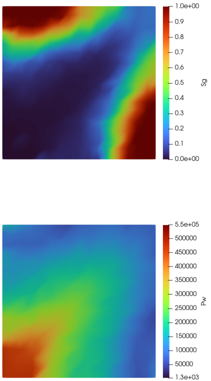


Test 2: Non-uniform anisotropic

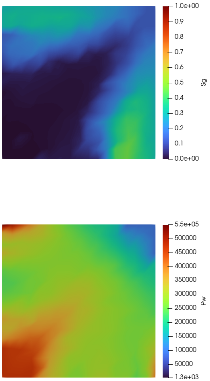
$$\Lambda(x,y) = \frac{0.15 \times 10^{-10}}{x^2+y^2} \begin{bmatrix} 0.1x^2+y^2 & 0.9xy \\ 0.9xy & x^2+0.1y^2 \end{bmatrix} [m^2], \quad \forall (x,y) \in \Omega. \quad t_f = 120\ s, \delta t = 0.005\ s.$$



Gas saturation [-] (up) and water pressure [Pa] (down) at $t = 10\ s$

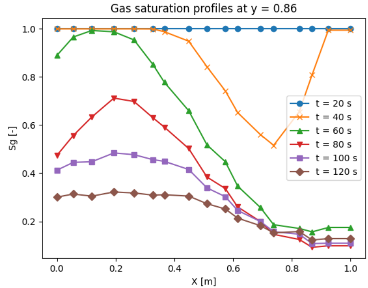


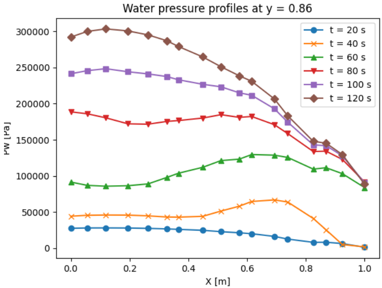
Gas saturation [-] (up) and water pressure [Pa] (down) at $t = 60\ s$

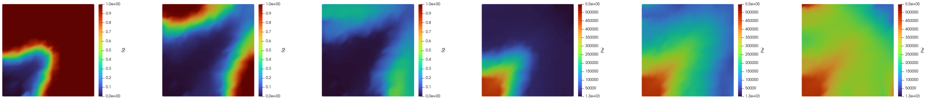


Gas saturation [-] (up) and water pressure [Pa] (down) at $t = 120\ s$

Test 2: profiles







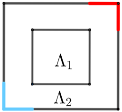
Test 3: Strong heterogeneity, m-DDFV method.

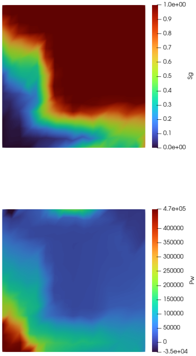
[Boyer and Hubert, 2008, Chainais-Hillairet et al, 2013.

$t_f = 150\ s, \quad \delta t = 0.001\ s.$

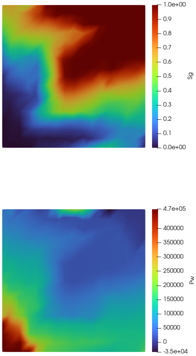
$$\Lambda_1(x,y) = 0.15 \times 10^{-10} \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix} [m^2],$$

$$\Lambda_2(x,y) = 0.15 \times 10^{-10} \times R_{\pi x} \times \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix} \times R_{\pi x}^{-1} [m^2].$$

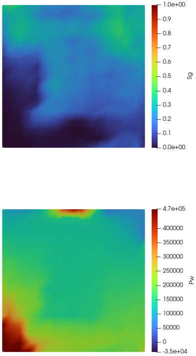




Gas saturation [-] (up) and water pressure [Pa] (down) at $t = 30\ s$

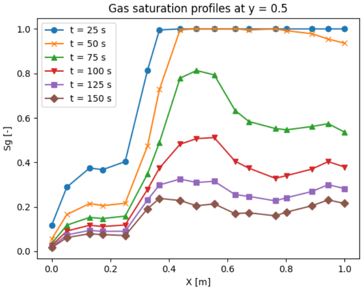


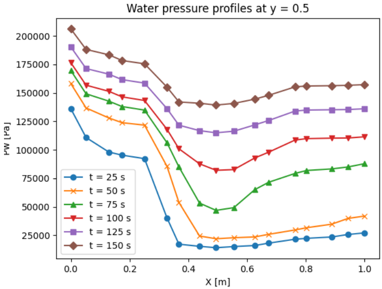
Gas saturation [-] (up) and water pressure [Pa] (down) at $t = 60\ s$

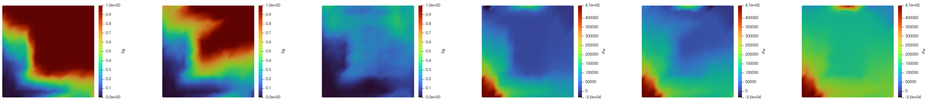


Gas saturation [-] (up) and water pressure [Pa] (down) at $t = 150\ s$

Test 3: profiles







DDFV numerical scheme

- maximum principle on the saturations,
- energy estimates,
- convergence to a weak solution up to a subsequence.

Thank you for your attention !!!

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