Bidomain Model Coupled To A Realistic Multi-Electrode Device In Cardiac Electrophysiology

SimCardioTest European Project

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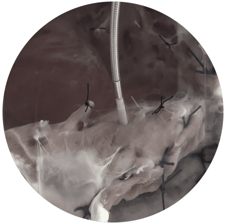
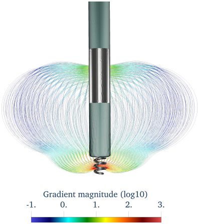
Context & Motivation

WP2 of SimCardioTest focuses on Pacing Leads and Catheters, especially on **energy required to trigger a cardiac beat** with a pacemaker.

(a) BOREA™pacemaker



(b) Lead implanted in the septum

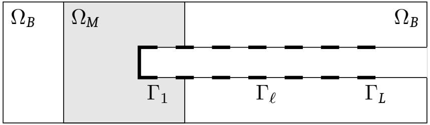


(c) Simulation of electrical field

Problematic

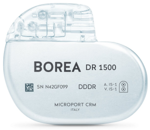
Electrophysiological propagation models already exist, **but we need to describe the stimulation**

- Three discontinuous parts by stimulation $\Rightarrow$ **Device circuit must be modeled**
  `Structural analysis of electric circuits and consequences for MNA`,
  D. Estévez Schwarz, C. Tischendorf, 2000
- Presence of a reactive layer at electrode boundaries $\Rightarrow$ **Layer must be modeled too**
  `Existence and Uniqueness for Electrode Models for Electric Current Computed Tomography`,
  E. Somersalo, M. Cheney and D. Isaacson, 1992



(a) Geometric setup

+ Contacts +



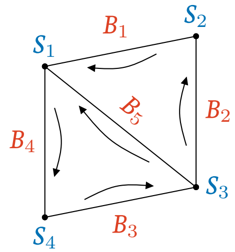
(b) BOREA™

Principle & Notations – I

- An **oriented graph** $\mathcal{G} = (S, B)$ with
  - N_n nodes,
  - N_b branches (dipoles)
- **Potentials** U in Volt and **currents** J in Ampere
- **Incidence matrix** $A = (a_{s,b})_{s\in S, b\in B}$ of size $N_\mathrm{n} \times N_\mathrm{b}$

$$a_{s,b} = \begin{cases} -1 & \text{if } s \text{ is an end point of } b \\ +1 & \text{if } s \text{ is a start point of } b \\ 0 & \text{elsewhere} \end{cases}$$

- Kirchhoff's **current law** $AJ = 0$ and **voltages** $V := A^\intercal U$.



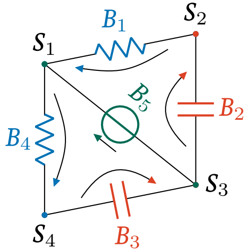
$$U = (U_{S_1}, U_{S_2}, U_{S_3}, U_{S_4})$$
$$J = (J_{B_1}, J_{B_2}, J_{B_3}, J_{B_4}, J_{B_5})$$

	B_1	B_2	B_3	B_4	B_5
s_1	1	0	0	−1	1
s_2	−1	1	0	0	0
s_3	0	−1	1	0	−1
s_4	0	0	−1	1	0

Principle & Notations – II

- Only contains **capacitors, resistors and voltage sources**
- **Split** graph and matrix accordingly:
  $B = B_c \cup B_g \cup B_s$ and $A = [A_c, A_g, A_s]$
- $AJ = 0 \Rightarrow \boxed{A_c J_c + A_g J_g + A_s J_s = 0}$
- $V = A^\intercal U \Rightarrow V_c = A_c^\intercal U, V_g = A_g^\intercal U$ and $V_s(t) = A_s^\intercal U$
- **Dipole laws**: $J_c = C\frac{dV_c}{dt} = C\frac{dA_c^\intercal U}{dt}$ and $J_g = GV_g = GA_g^\intercal U$
- **Differential algebraic equations (DAE)**

(U, J_s) solution of $\boxed{\begin{cases} A_c C \frac{dA_c^\intercal U}{dt} + A_g G A_g^\intercal U + A_s J_s = 0 \\ A_s^\intercal U = V_s \end{cases}}$



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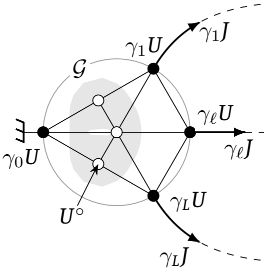
Generic Models

Multi-Electrode Device Model

- Graph (S, B) oriented with $L+1$ **specialized nodes**
- **Potentials** split as $U = (U^\circ, \gamma U)$ of size N_n, internal potentials U°
- **Dirichlet boundary condition**, $\gamma_0 U = 0$ (ground)
- **Coupling boundary potentials** $\gamma_1 U, \ldots, \gamma_L U$
- **Coupling output currents** $\gamma J = (\gamma_1 J, \ldots, \gamma_L J)$



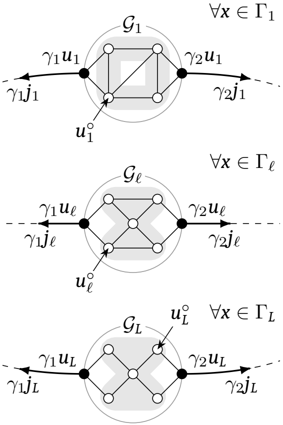
(U, J_s) solution of DAE

$$\begin{cases} A_c C \frac{dA_c^\intercal U}{dt} + A_g G A_g^\intercal U + A_s J_s + \gamma A \gamma J = 0 \\ A_s^\intercal U = V_s \end{cases}$$

with $\gamma A := [0, \mathrm{Id}_L]^\intercal$.

Generic Models

Equivalent Models for each electrode



On each electrode $\ell = 1, \dots, L$ (without voltage sources)

- **Defined spatially:** $\forall x \in \Gamma_\ell$
- **Potentials** split as $u_\ell = (u_\ell^\circ, \gamma u_\ell)$ of size N_ℓ
- **Coupling boundary potentials** $\gamma_1 u_\ell, \gamma_2 u_\ell$
- **Coupling output currents** $\gamma j_\ell = (\gamma_1 j_\ell, \gamma_2 j_\ell)$

u_ℓ solution of x-ODE

$$\boxed{A_{\ell,c} C_\ell \frac{\partial A_{\ell,c}^\top u_\ell}{\partial t} + A_{\ell,g} G_\ell A_{\ell,g}^\top u_\ell + \gamma A_\ell \gamma j_\ell = 0}$$

on Γ_ℓ

with $\gamma A_\ell := [0, \mathrm{Id}_2]^\top$.

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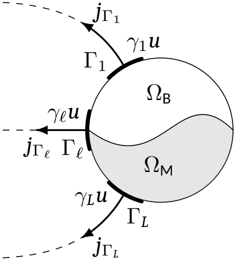
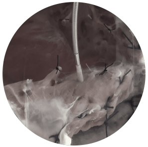
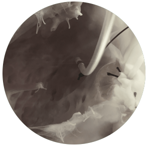
Generic Models

Bidomain Model (PDE)

- **Tissue** Ω_{M} – intracellular and extracellular
- **Bath/Blood** Ω_{B} – passive media
- **Transmembrane voltage** $[u] = u_{\mathrm{i}} - u$
- **Conductivity tensors** σ and σ_{i}
- L **electrodes** at boundary $\bigcup_{\ell=1}^{L} \Gamma_\ell \subset \partial\left(\overline{\Omega_{\mathrm{M}}} \cup \overline{\Omega_{\mathrm{B}}}\right)$
- $\gamma_\ell u = u_{|\Gamma_\ell}$, and $-\sigma \nabla u \cdot \mathbf{n} = j_{\Gamma_\ell}$ on Γ_ℓ







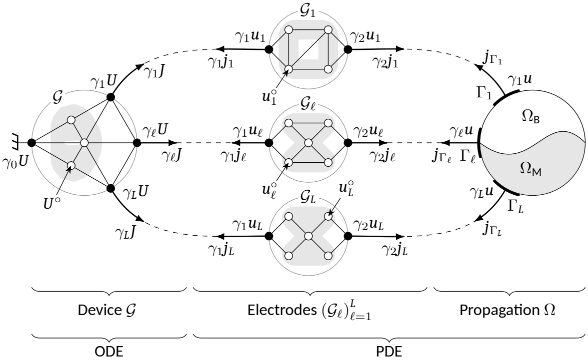
(u, u_{i}) solution of PDE

$$-\operatorname{div}(\sigma_{\mathrm{i}} \nabla u_{\mathrm{i}}) = -j_{\mathrm{m}} \text{ in } \Omega_{\mathrm{M}} \text{ and } \begin{cases} -\operatorname{div}(\sigma \nabla u) = j_{\mathrm{m}} & \text{in } \Omega_{\mathrm{M}} \\ -\operatorname{div}(\sigma \nabla u) = 0 & \text{in } \Omega_{\mathrm{B}} \end{cases}$$

with membrane current density $j_{\mathrm{m}} = \chi\left(c_{\mathrm{m}} \partial_t [u] + I_{\mathrm{ion}}([u], h, t)\right)$, and Ionic Model on h.

Generic Models

Transmission conditions



— **Potentials**

$\gamma_\ell U = \gamma_1 u_\ell(x)$, for $x \in \Gamma_\ell$

$\gamma_2 u_\ell(x) = \gamma_\ell u(x)$, for $x \in \Gamma_\ell$

— **Currents**

$\gamma_\ell J + \int_{\Gamma_\ell} \gamma_1 j_\ell(x) = 0$

$\gamma_2 j_\ell(x) + j_{\Gamma_\ell}(x) = 0$, for $x \in \Gamma_\ell$

Sketch of the proof – I

Global variational formulation

For all $\boldsymbol{\phi} = (\phi, \phi_{\mathrm{i}}, \phi_1, \ldots, \phi_L, \Phi) \in C_c^\infty$ of $[0,T) \times \left\{\overline{\Omega}, \overline{\Omega_{\mathrm{M}}}, \left(\overline{\Gamma_\ell}\right)^{N_\ell}, \mathbb{R}^{N_{\mathrm{n}}}\right\}$,
find $\boldsymbol{u} := (u, u_{\mathrm{i}}, u_1, \ldots, u_L, U)$ such that $\gamma_0 U = 0$ + transmission conditions on potentials, and J_{s}, verifying

$$\begin{cases} \displaystyle\int_0^T \int_{\Omega_{\mathrm{M}}} \sigma_{\mathrm{i}} \nabla u_{\mathrm{i}} \cdot \nabla \phi_{\mathrm{i}} + \int_0^T \int_{\Omega} \sigma \nabla u \cdot \nabla \phi + \int_0^T \int_{\Omega_{\mathrm{M}}} \left(c_{\mathrm{m}} \frac{\partial [u]}{\partial t} [\phi] + f\left([u], h, t\right) [\phi] \right) \\ \displaystyle\qquad + \int_0^T \left(C \frac{\mathrm{d} A_{\mathrm{c}}^{\mathsf{T}} U}{\mathrm{d}t} \cdot A_{\mathrm{c}}^{\mathsf{T}} \Phi + G A_{\mathrm{g}}^{\mathsf{T}} U \cdot A_{\mathrm{g}}^{\mathsf{T}} \Phi + A_{\mathrm{s}} J_{\mathrm{s}} \cdot \Phi \right) \\ \displaystyle\qquad + \sum_{\ell=1}^{L} \int_0^T \int_{\Gamma_\ell} \left(C_\ell \frac{\partial A_{\ell,\mathrm{c}}^{\mathsf{T}} u_\ell}{\partial t} \cdot A_{\ell,\mathrm{c}}^{\mathsf{T}} \phi_\ell + G_\ell A_{\ell,\mathrm{g}}^{\mathsf{T}} u_\ell \cdot A_{\ell,\mathrm{g}}^{\mathsf{T}} \phi_\ell \right) = 0, \\ A_{\mathrm{s}}^{\mathsf{T}} U = V_{\mathrm{s}} \end{cases}$$

+ IPP $\Rightarrow$ weak solution

Sketch of the proof – II

Global variational formulation semi-discretized in time

Find $\boldsymbol{u}^n := (u^n, u_\mathrm{i}^n, u_1^n, \dots, u_L^n, U^n) \in \mathcal{H}_0$, and $J_s^n \in \mathbb{R}^{N_s}$ with

$$\mathcal{H} := H^1(\Omega_\mathrm{M}) \times H^1(\Omega) \times \Pi_{\ell=1}^{L} L^2(\Gamma_\ell)^{N_\ell} \times \mathbb{R}^{N_\mathrm{n}},$$

$$\mathcal{H}_0 := \{\boldsymbol{u} \in \mathcal{H} \text{ s.t. } \boxed{\gamma_0 U = 0, \gamma_\ell U = \gamma_1 \boldsymbol{u}_\ell, \gamma \boldsymbol{u} = \gamma_2 \boldsymbol{u}_\ell}, \ell = 1, \dots, L\},$$

verifying, for all $\boldsymbol{\phi} := (\phi, \phi_\mathrm{i}, \phi_1, \dots, \phi_L, \Phi) \in \mathcal{H}_0$,

$$a\left(\frac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{\Delta t}, \boldsymbol{\phi}\right) + b(\boldsymbol{u}^n, \boldsymbol{\phi}) + (f([u^n]), [\phi])_{\Omega_\mathrm{M}} + A_\mathrm{s} J_\mathrm{s}^n \cdot \Phi = 0,$$

$$A_\mathrm{s}^\intercal U^n = V_\mathrm{s}^n$$

$$a(\delta \boldsymbol{u}, \boldsymbol{\phi}) = (c_\mathrm{m}[\delta u], [\phi])_{\Omega_\mathrm{M}} + \sum_{\ell=1}^{L} \left(C_\ell A_{\ell,\mathrm{c}}^\intercal \delta u_\ell, A_{\ell,\mathrm{c}}^\intercal \phi_\ell\right)_{\Gamma_\ell} + C A_\mathrm{c}^\intercal \delta U \cdot A_\mathrm{c}^\intercal \Phi,$$

$$b(\boldsymbol{u}, \boldsymbol{\phi}) = (\sigma_\mathrm{i} \nabla u_\mathrm{i}, \nabla \phi_\mathrm{i})_{\Omega_\mathrm{M}} + (\sigma \nabla u, \nabla \phi)_\Omega + \sum_{\ell=1}^{L} \left(G_\ell A_{\ell,\mathrm{g}}^\intercal u_\ell, A_{\ell,\mathrm{g}}^\intercal \phi_\ell\right)_{\Gamma_\ell} + G A_\mathrm{g}^\intercal U \cdot A_\mathrm{g}^\intercal \Phi.$$

Sketch of the proof – III

Existence

— **Hypothesis**

A_s has full column rank

— **Consequence**

$\ker(A_s) = \{0\}$ and $\mathrm{Im}\,(A_s^\intercal) = \ker(A_s)^\perp = \mathbb{R}^{N_s}$, $\Rightarrow \forall V_s^n \in \mathbb{R}^{N_s}$ there exists $U_0^n \in \mathbb{R}^{N_n}$ such as $A_s^\intercal U_0^n = V_s^n$ and $\|\boldsymbol{u}_0^n\|_{\mathcal{H}} \leq cst\,\|V_s^n\|_{\mathbb{R}^{N_n}}$.

— **Variational formulation in the constraint space**

Find $\boldsymbol{u}_\star^n \in \mathcal{H}_0^{\ker}$, with $\boldsymbol{u}^n = \boldsymbol{u}_\star^n + \boldsymbol{u}_0^n$ and $\boxed{\mathcal{H}_0^{\ker} := \{\boldsymbol{u} \in \mathcal{H}_0 \mid U \in \ker(A_s^\intercal)\}}$, verifying for all $\phi_\star \in \mathcal{H}_0^{\ker}$,

$$a\left(\frac{\boldsymbol{u}_\star^n + \boldsymbol{u}_0^n - \boldsymbol{u}^{n-1}}{\Delta t}, \phi_\star\right) - b\left(\boldsymbol{u}_\star^n + \boldsymbol{u}_0^n, \phi_\star\right) + \left(f\left([u^n]\right), [\phi]\right)_{\Omega_M} = 0$$

Lax-Milgram and Banach fixed-point theorems: existence of $\boldsymbol{u}_\star^n$ in $\mathcal{H}_0^{\ker}$.

— **Resolution of J_s^n equation in $\mathbb{R}^{N_s}$**

- $\forall\{e_k\}$ basis of $\mathbb{R}^{N_s}$, $\exists\{\phi_k\} \subset \mathcal{H}_0$ s.t. $e_k = A_s^\intercal \Phi_k$, and J_s^n solves the linear systems $J_s^n \cdot e_k = w_k$,
- w_k are independent of the choice of $\{\phi_k\}$.

Sketch of the proof – IV

Compactness & passing to the limit

We have a sequence $(\boldsymbol{u}_0^n)$ bounded by data, and $(\boldsymbol{u}_\star^n)$ in $\mathcal{H}_0^{\text{ker}}$ with *a priori* estimates.

— ***A priori* estimates on**

$$a\left(\boldsymbol{u}^n, \boldsymbol{u}^n\right) \qquad \sum_n \Delta t b\left(\boldsymbol{u}^n, \boldsymbol{u}^n\right) \qquad \sum_n \Delta t a\left(\frac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{\Delta t}, \frac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{\Delta t}\right)$$

for compactness (as f is nonlinear)

— **Compactness**: discrete Aubin-Simon (Gallouët-Latché theorem)
$\Rightarrow$ up to subsequence $\boldsymbol{u}^n \to \boldsymbol{u}$ (strong convergence)

— **Passing to the limit**
$\Rightarrow$ $\boldsymbol{u}$ solution of the continuous problem.

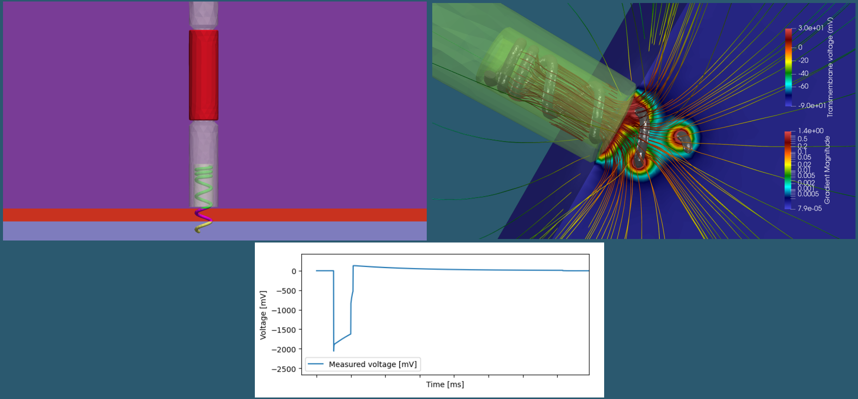


Figure: Pacemaker + 2 R//C contacts + geometry inspired of a standard VEGA lead