Schéma numérique décalé pour la simulation d'écoulements à bas nombre de Mach

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Outline

Low-Mach number limit problem

For the sake of simplicity we start with Euler barotropic equations, defined in $\Omega \subset \mathbb{R}^d \times [0, T]$:

$$\begin{cases} \partial_{\tilde{t}}\tilde{\rho} + \nabla_{\tilde{x}} \cdot (\tilde{\rho}\tilde{\mathbf{u}}) = 0 \\ \partial_{\tilde{t}}(\tilde{\rho}\tilde{\mathbf{u}}) + \nabla_{\tilde{x}} \cdot (\tilde{\rho}\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}}) + \dfrac{\nabla_{\tilde{x}}\tilde{p}}{\gamma \mathbf{M}^2} = 0 \end{cases} \quad (1)$$

with

- ρ is the density
- equation of state $p = f(\rho)$, p is the pressure
- $\mathbf{u}$ is the the velocity
- Let x_0 characteristic length, t_0 its characteristic time, ρ_0 characteristic density. $u_0 = x_0/t_0$ characteristic speed $c_0^2 = p'(\rho_0)$ characteristic sound velocity . The Mach number $\mathbf{M} := \dfrac{u_0}{c_0}$. (with $\gamma = \tilde{p}'(1)$):

$\Longrightarrow$ Singular limit

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The continuous case I

Let

- $\tau = \tilde{t}/\mathbf{M}$ acoustic time scale.
- Asymptotic expansion in Mach number $\mathbf{M}$:
  $\varphi(x,t,\tau,\mathbf{M}) = \varphi(x,t,\tau)^{(0)} + \mathbf{M}\varphi(x,t,\tau)^{(1)} + \mathbf{M}^2\varphi(x,t,\tau)^{(2)} + \mathcal{O}(\mathbf{M}^3)$.

We obtain ([JP22], [Mül98]):

$$\begin{cases} \partial_\tau \tilde{\rho}^{(1)} + \nabla \cdot (\tilde{\rho}^{(0)} \tilde{\mathbf{u}}^{(0)}) = -\dfrac{d}{d\tilde{t}} \tilde{\rho}^{(0)} \\ \partial_\tau (\tilde{\rho}^{(0)} \tilde{\mathbf{u}}^{(0)}) + \tilde{c}^2(\tilde{\rho}^{(0)}) \nabla \tilde{\rho}^{(1)} = 0 \end{cases} \tag{2}$$

The continuous case II

In other words, if we let $\tilde{\mathbf{u}}^{(0)} = \tilde{\mathbf{u}}_{\Psi}^{(0)} + \tilde{\mathbf{u}}_{\varphi}^{(0)}$

- $\nabla \times \tilde{\mathbf{u}}_{\varphi}^{(0)} = 0$
- $\nabla \cdot \tilde{\mathbf{u}}_{\Psi}^{(0)} = 0.$

$$\begin{pmatrix} \tilde{\rho} \\ \tilde{\mathbf{u}} \end{pmatrix} = \underbrace{\begin{pmatrix} \tilde{\rho}^{(0)} \\ \tilde{\mathbf{u}}_{\Psi}^{(0)} \end{pmatrix} (\tilde{x}, \tilde{t})}_{\text{incompressible part}} + \underbrace{\begin{pmatrix} \mathbf{M}\tilde{\rho}^{(1)} \\ \tilde{\mathbf{u}}_{\varphi}^{(0)} \end{pmatrix} (\tilde{x}, \tilde{t}, \frac{\tilde{t}}{\mathbf{M}})}_{\text{acoustic part}} + \begin{pmatrix} \mathcal{O}(\mathbf{M}^2) \\ \mathcal{O}(\mathbf{M}) \end{pmatrix}$$

Strong convergence to incompressible limit if

$$\tilde{\mathbf{u}}_0 = \tilde{\mathbf{u}}_0^{(0)} + \mathcal{O}(M), \nabla \cdot \tilde{\mathbf{u}}_0^{(0)} = 0 \quad \text{and} \quad \tilde{\rho}_0 = \tilde{\rho}^{(0)} + \mathcal{O}(M^2), \tilde{\rho}^{(0)} \in \mathbb{R}$$

Those conditions = "well-prepared" (in periodic boundary conditions,see [KM81], [Sch94] or [GM04] for explanation on linear model)

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The discrete case I : collocated schemes

ρ, **u** at the center of the cell
Dependency on the mesh (see [DOR10] [Rie08] [Del10], [GN17])

- Godunov scheme : solving the Riemann problem at each interface introduce parasite acoustic waves even for well-prepared data [GM04]
- Roe scheme in quad : the stationary space is too small ([Del10][DOR10][Rie08])

Colocalized schemes : Corrections ?

Pressure centered scheme (Roe-Dellach) : consistent with wave system **but** admits checkerboard modes [Del09]

$$\begin{cases} \partial_\tau \tilde{\rho}_i^{(1)} + \frac{1}{|\tilde{K}_i|} \sum_{j\in\nu(i)} |\tilde{\Gamma}_{i,j}| \left(\{\{\rho\tilde{\mathbf{u}}\}\}^{(0)} \cdot \mathbf{n}_{ij} - \frac{\tilde{a}_{ij}^{(0)}}{2} [[\tilde{p}^{(1)}]] \right) = -\frac{d}{d\tilde{t}} \tilde{\rho}^{(0)} \\ \partial_\tau (\rho\tilde{\mathbf{u}}^{(0)})_i + \frac{1}{|\tilde{K}_i|} \sum_{j\in\nu(i)} |\tilde{\Gamma}_{i,j}| \{\{ \frac{\tilde{p}^{(1)}}{\gamma} \}\} \mathbf{n}_{ij} = 0 \end{cases}$$

Colocalised schemes : the not-so-mysterious triangle case

In triangles, better behaviour with respect to Low-Mach flows.

- For all i, j $[\![\mathbf{u} \cdot \mathbf{n}]\!]_{i,j} = 0$. For each face (choosing an arbitrary orientation) we can correspond a unique value $\Longleftrightarrow \exists ! U_f \in \mathbb{R}^F$, and (see [Nic92])

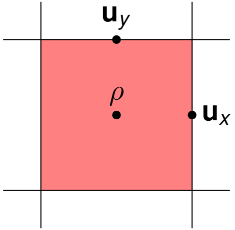
$$U_f = rot_h \Psi \quad \text{with} \quad \Psi \in \mathbb{P}^1(\Omega)$$

  $rot_h \in \mathbb{R}^{V \times F}$ is a "discrete rotationnal"

- Triangles : in each cell we can reconstruct a **unique** constant velocity vector from the face values (see [Gui09], [Nic92]) $\longrightarrow$ this analysis does not stand in quads

The discrete case II : staggered schemes

ρ at the center of the cell , **u** at the center of the face.



2 dof for 2 components $\equiv$ "unisolvence"

MAC first introduced for incompressible flows by [Har65] .
Compared to its first apparition, recent explosion in the research of an "all-Mach" scheme (for example [HLN13b] [HKL14] [BHL22] [BLW08], [HLS21])

Staggered and de Rham complexes I

Staggered scheme seems to preserve continuous structures at the discrete level such as :
de Rham complexes, sequences of the type

$$0 \longrightarrow H^1(\Omega) \xrightarrow{\nabla} H(rot;\Omega) \xrightarrow{rot} H(div;\Omega) \xrightarrow{div} L^2(\Omega) \to 0$$

but **discrete**

Staggered and De Rham complexes II

Important byproducts of de Rham complexes :

- Hodge decomposition

$$\mathbf{u}_h = \mathbf{u}_\Psi + \mathbf{u}_\varphi$$

- Discrete grad, div duality, for some scalar product $(\nabla_h p, \mathbf{u}) = (p, -\nabla_h \cdot \mathbf{u})$
- if Ω is of trivial topology : exactness of complex. This propriety is equivalent to the surjectivity of $\nabla\cdot$ (inf-sup stability)

Last 2 propreties seem important to prove convergence to incompressible schemes [HLS21]

Outline

Starting the other way around : from the wave system to Euler

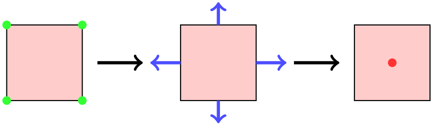
Our "algorithm" to obtain low Mach scheme is the following :

1) Formally the low Mach number behaviour of a numerical scheme is encapsulated in the following wave system :

$$\begin{cases} \partial_\tau p + \frac{1}{\rho} div \mathbf{u} = 0 \\ \partial_\tau \mathbf{u} + \kappa \nabla p = 0 \\ p_{|\partial\Omega} = p_b \\ \mathbf{u} \cdot \mathbf{n}_{|\partial\Omega} = \mathbf{u}_b \cdot \mathbf{n} \end{cases}$$

2) To ensure those good properties $\implies$ base discretization on Nédélec-Raviart-Thomas de Rham complex (for trivial topologies again)([EG04] or [Arn18] for abstract point of view): in 2D

$$\{0\} \xrightarrow{id} c\mathbb{Q}^1(\Omega) \xrightarrow{\nabla}{}^{\perp} \mathbb{RT}^1(\Omega) \xrightarrow{\nabla\cdot} d\mathbb{Q}^0(\Omega) \xrightarrow{0} 0$$



From FE to FV : the numerical scheme

Using a mass-lumping we obtain the following numerical scheme :

$$\begin{cases} |K|\partial_\tau p_K + \dfrac{1}{\rho}\displaystyle\sum_{\sigma\subset\partial K}|\sigma|\varepsilon_K(\sigma)\mathbf{u}_\sigma = \theta\dfrac{c}{2}\displaystyle\sum_{\sigma\subset\partial K}|\sigma|[\![\, p\,]\!]_\sigma \\ \\ |D_\sigma|\partial_\tau \mathbf{u}_\sigma + \kappa|\sigma|[\![\, p\,]\!]_\sigma = \theta\dfrac{c}{2}|\sigma|[\![\, \widetilde{div\mathbf{u}}\,]\!]_\sigma \\ \\ \mathbf{u}_\sigma = \mathbf{u}_b\cdot\mathbf{n}_\sigma \;\; \forall\sigma\in\mathcal{F}^{ext} \\ p_K = (p_b)_K \;\; \forall K\in\mathcal{M}^{ext} \end{cases}$$

where $|K|$ primal volume, $|D_\sigma|$ dual volume associated to a face σ, $|\sigma|$ length of the face , $[\![\, p\,]\!]_\sigma := p_K - p_L$ and $(\widetilde{div\mathbf{u}})_K := \dfrac{1}{|\partial K|}\displaystyle\sum_{\sigma\subset\partial K}|\sigma|\varepsilon_K(\sigma)\mathbf{u}_\sigma$.

$\theta = 0$ Euler implicit $\theta = 1$ in Euler explicit

Outline

Energy dissipation

Theorem

Dissipation of the following natural energy $\sum_{K\subset\mathcal{M}}|K|\rho_K\frac{p_K^2}{2}+\sum_{\sigma\subset\mathcal{F}^{int}}|D_\sigma|\frac{\boldsymbol{u}_\sigma^2}{2}$.

- *In Euler implicit time integration, unconditionally*
- *In Euler explicit under the following CFL condition*

$$\frac{c\delta\tau\, max(|\partial K|)}{min(|K|,|D_\sigma|)}\leq\frac{1}{2+\nu_{max}}$$

where $\nu_{max}=4$ *in quads,* $\nu_{max}=3$ *in triangles*

Existence of long time limit I

Theorem

Let $\Omega \subset \mathbb{R}^3$ *an open domain regular enough* $\boldsymbol{u}_0, \boldsymbol{u}_b \in \mathbb{RT}^1(\Omega)$ *such that* $\int_{\partial\Omega} \boldsymbol{u}_b \cdot \boldsymbol{n} d\Gamma = 0$*.Then we have the following Hodge-Decomposition*

$$\boldsymbol{u}_0 = (\boldsymbol{u}_0)_\varphi + (\boldsymbol{u}_0)_\Psi$$

where $div((\boldsymbol{u}_0)_\Psi) = 0$, $(\boldsymbol{u}_0)_\Psi \cdot \boldsymbol{n}_{|\partial\Omega} = \boldsymbol{u}_b \cdot \boldsymbol{n}_{|\partial\Omega}$

This comes "naturally" from the use of complexes

Existence of long time limit II

Using those dissipation properties we can prove the following

Theorem

Let

- $p^{\infty} = p_b$
- $\boldsymbol{u}^{\infty} = (\boldsymbol{u}_0)_{\Psi} \in \mathbb{RT}^1(\Omega)$ *such that* $div(\boldsymbol{u}_h^{\infty}) = 0$ *and* $\boldsymbol{u}^{\infty} \cdot \boldsymbol{n}_{|\partial\Omega} = \boldsymbol{u}_b \cdot \boldsymbol{n}_{|\partial\Omega}$

Then every solution built with the previous schemes converges in long time to this limit

Proof is based on the fact that $(p^{\infty}, \mathbf{u}^{\infty})$ is a stationnary solution of the discrete wave system.

$$div(\mathbf{u}^{\infty}) = 0 \nRightarrow \vec{\Delta}\mathbf{u}^{\infty} = 0$$

Numerical results I

C++ code Solverlab, results from simulation on cylinder with $n_r = 5$ and $n_\theta = 16$.

$p_0 = 1.5$, $p_b = 2$, $\mathbf{u}_0 = (0,0)^t$, $u_b = (1,0)^t$

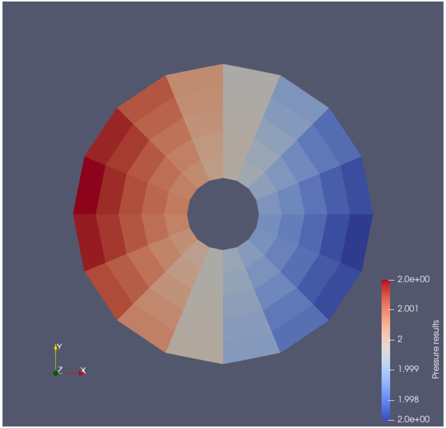


Figure 1: p^∞

Numerical results II

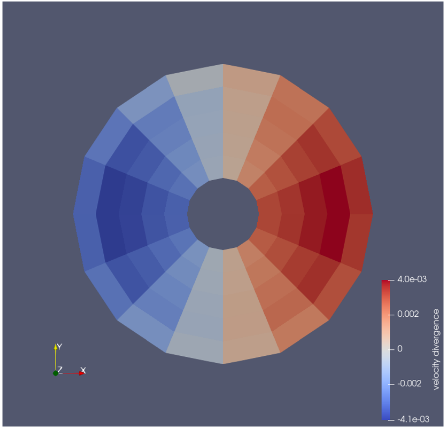


Figure 2: $div(\mathbf{u}^{\infty})$

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Conclusions and perspectives

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- Energy dissipation for Euler implicit and explicit time integrations
- Theoretical analysis of the convergence in long time limit
- Analysis is valid in quad and triangles
- We infer from our analysis that using staggered schemes do not imply automatically the precision at Low Mach (Euler explicit time integration : non-classical grad div stabilization needed in order to both obtain energy dissipation and preservation of stationnary states)

Perspectives:

$\longrightarrow$ Extension of the scheme on Euler barotropic and full Euler

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$\rightarrow$ Extension of the scheme on Euler barotropic and full Euler

Questions?

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