A proof of Marchal's conjecture: from the Lagrange triangle to the figure eight

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In the N-body problem, *choreographies* are periodic solutions where N equal masses follow each other along a closed curve. Each mass takes periodically the position of the next after a fixed interval of time. In 1993 [3], Moore discovers numerically a choreography for $N = 3$ in the shape of an *eight*. The proof of its existence is established in 2000 by Chenciner and Montgomery [1]. In the same year [2], Marchal publishes his work on the most symmetric family of spatial periodic orbits, leaving the center manifold of the Lagrange triangle by continuation with respect to the period. This continuation class is referred to as the P_{12} family. Noting that the figure eight possesses the same twelve symmetries than the P_{12} family, the author claims that it ought to belong to P_{12}. This is known as *Marchal's conjecture*.

In this talk, we present a constructive proof of Marchal's conjecture. We formulate a one parameter family of functional equations, whose zeros correspond to periodic solutions satisfying the symmetries of the P_{12} family ; the frequency of a rotating frame is used as the continuation parameter. The goal is then to prove the uniform contraction of a mapping, in a neighbourhood of an approximation of the family of choreographies starting at the Lagrange triangle and ending at the figure eight. The contraction is set in the Banach space of rapidly decaying Fourier-Chebyshev series coefficients. While the Fourier basis is employed to model the temporal periodicity of the solutions, the Chebyshev basis captures their parameter dependence. In this framework, we obtain a high-order approximation of the family as a finite number of Fourier polynomials, where each coefficient is itself given by a finite number of Chebyshev polynomials. The contraction argument hinges on the local isolation of each individual choreography in the family. However, symmetry breaking bifurcations occur at the Lagrange triangle and the figure eight. At the figure eight, there is a translation invariance in the x-direction. We explore how the conservation of the linear momentum in x can be leveraged to impose a zero average value in time for the choreographies. Lastly, at the Lagrange triangle, its (planar) homothetic family meets the (off-plane) P_{12} family. We discuss how a blow-up (as in "zoom-in") method provides an auxiliary problem which only retains the desired P_{12} family.

[1] A. Chenciner, R. Montgomery. *A remarkable periodic solution of the three-body problem in the case of equal masses.* Annals of mathematics, **152**, 881–901, 2000. doi :10.2307/2661357.

[2] C. Marchal. *The three-body problem*, vol. 4 of *Studies in Astronautics.* Elsevier Science Publishers, B.V., Amsterdam, 1990. With a foreword by Victor Szebehely, With French, Russian, German, Spanish, Japanese, Chinese and Arabic summaries.

[3] C. Moore. *Braids in classical dynamics.* Physical Review Letters, **70**, 3675–3679, 1993. doi : 10.1103/PhysRevLett.70.3675.

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