Local density interpolation applied to boundary integral methods

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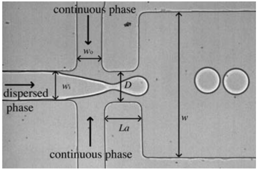
Introduction

(a) Euglenoid algae swimming by deforming its shape.



(b) Example of droplet production in microfluidics

Figure: Boundary integral equations (BIE) applied to deforming surfaces

Advantage:

- Less dimension, easier to mesh;
- Directly applicable to unbounded domains.

Boundary integral equations

- General formulation of elliptic equations

$$\begin{cases} \mathcal{L}u(\boldsymbol{r}) = 0 & \text{for } \boldsymbol{r} \in \Omega \text{ or } \mathbb{R}^d \backslash \overline{\Omega}, \\ \text{Boundary condition} & \text{for } \boldsymbol{x} \in \Gamma := \partial\Omega, \\ \text{Radiation condition at infinity} & \text{if } \boldsymbol{r} \in \mathbb{R}^d \backslash \overline{\Omega}, \end{cases}$$

- Examples

	2-d Laplace	3-d Helmholtz
$\mathcal{L}$	$-\Delta$	$-\Delta - k^2$
Green's function	$G(\boldsymbol{x},\boldsymbol{y}) = \frac{\log\lvert\boldsymbol{x}-\boldsymbol{y}\rvert}{2\pi}$	$G(\boldsymbol{x},\boldsymbol{y}) = \frac{e^{ik\lvert\boldsymbol{x}-\boldsymbol{y}\rvert}}{4\pi\lvert\boldsymbol{x}-\boldsymbol{y}\rvert}$
Boundary condition	$\gamma_0 u = u; \gamma_1 u = \partial_n u$	$\gamma_0 u = u; \gamma_1 u = \partial_n u$
Radiation condition	$u(\boldsymbol{r}) = O(\log\lvert\boldsymbol{r}\rvert)$	$\partial_n u - iku(\boldsymbol{r}) = o(\lvert\boldsymbol{r}\rvert^{-1})$

Boundary integral equations

Let G be the Green's function associated with $\mathcal{L}$.

- Ansatz: for $\boldsymbol{r} \in \Omega$ or $\mathbb{R}^d \backslash \overline{\Omega}$

$$u(\boldsymbol{r}) = \alpha \int_\Gamma \gamma_{1,\boldsymbol{y}} G(\boldsymbol{r}, \boldsymbol{y}) \varphi(\boldsymbol{y}) \mathrm{d}s_{\boldsymbol{y}} - \beta \int_\Gamma \gamma_{0,\boldsymbol{y}} G(\boldsymbol{r}, \boldsymbol{y}) \varphi(\boldsymbol{y}) \mathrm{d}s_{\boldsymbol{y}};$$

- Take the trace γ_0 on both sides we have: for $\boldsymbol{x} \in \Gamma$

$$u(\boldsymbol{x}) = \pm\alpha \frac{\varphi(\boldsymbol{x})}{2} + \alpha \int_\Gamma \gamma_{1,\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y}) \varphi(\boldsymbol{y}) \mathrm{d}s_{\boldsymbol{y}} - \beta \int_\Gamma \gamma_{0,\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y}) \varphi(\boldsymbol{y}) \mathrm{d}s_{\boldsymbol{y}};$$

- Integral operators: for $\boldsymbol{x} \in \Gamma$

$$\text{Single-layer: } S[\varphi](\boldsymbol{x}) := \int_\Gamma \gamma_{0,\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y}) \varphi(\boldsymbol{y}) \mathrm{d}s(\boldsymbol{y}),$$

$$\text{Double-layer: } K[\varphi](\boldsymbol{x}) := \text{ p.v. } \int_\Gamma \gamma_{1,\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y}) \varphi(\boldsymbol{y}) \mathrm{d}s(\boldsymbol{y}).$$

Difficulty: *singular* integrals on surfaces.

Nyström method

Target: approach the integral

$$\alpha K[\varphi](\boldsymbol{x}) - \beta S[\varphi](\boldsymbol{x}) = \int_\Gamma \left(\alpha \gamma_{1,\boldsymbol{y}} G(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y}) - \beta G(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y})\right) \mathrm{d}s_{\boldsymbol{y}}$$

by nodal values of φ at $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_N \in \Gamma$.
If regular

$$\alpha K[\varphi](\boldsymbol{x}) - \beta S[\varphi](\boldsymbol{x}) \approx \sum_{i=1}^{N} \left(\alpha \gamma_1 G(\boldsymbol{x},\boldsymbol{y}_i) - \beta G(\boldsymbol{x},\boldsymbol{y}_i)\right)\varphi(\boldsymbol{y}_i) w_i.$$

Needs correction near $\boldsymbol{x}$.

Local density interpolation method (DIM)

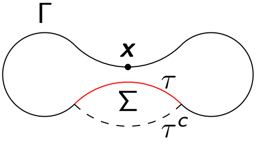


Figure: Using Green's identity, transport the integral to a surface farther away from the source point.

Construct a domain Σ such that $\tau \subset \partial\Sigma$, and a function Φ satisfying

$$\begin{cases} \mathcal{L}\Phi = 0, & \text{in } \Sigma; \\ \gamma_0\Phi = \alpha\varphi, & \text{on } \tau; \\ \gamma_1\Phi = \beta\varphi, & \text{on } \tau, \end{cases} \tag{1}$$

so that

$$\frac{\varphi(\boldsymbol{x})}{2}1_{\boldsymbol{x}\in\tau} + \int_\tau \left(\alpha\gamma_{1,\boldsymbol{y}}G(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y}) - \beta G(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y})\right)\mathrm{d}s_{\boldsymbol{y}}$$
$$= \int_{\tau^c} \left(\gamma_{1,\boldsymbol{y}}G(\boldsymbol{x},\boldsymbol{y})\Phi(\boldsymbol{y}) - G(\boldsymbol{x},\boldsymbol{y})\gamma_1\Phi(\boldsymbol{y})\right)\mathrm{d}s_{\boldsymbol{y}}.$$

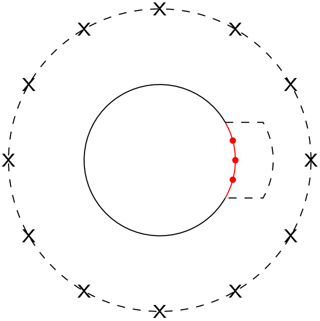
Error analysis

With the nodal values on τ, we construct Φ as follows:

$$\varphi \xrightarrow[E_{\tau}^{\text{interp}}]{\text{interpolation}} \tilde{\varphi} \xrightarrow{\text{extension}} \widetilde{\Phi} \xrightarrow[E_{\tau^c}^{\text{quad}}]{\text{quadrature}} Q_{\tau^c}(\widetilde{\Phi}).$$

$$E_{\tau} \leq E_{\tau}^{\text{interp}} + E_{\tau^c}^{\text{quad}},$$
$$E_{\tau}^{\text{interp}} \lesssim |\tau| \|\varphi - \tilde{\varphi}\|_{L_{\tau}^{\infty}},$$
$$E_{\tau^c}^{\text{quad}} \to \text{dependent on } \operatorname{dist}(\boldsymbol{x}, \tau^c).$$

Construction using source points



$$\widetilde{\Phi}(\boldsymbol{r}) := \sum_{\ell=1}^{L} G(\boldsymbol{r}; \boldsymbol{z}_\ell) c_\ell.$$

$$\gamma_0 \widetilde{\Phi}(\boldsymbol{y}_j) = \alpha \varphi(\boldsymbol{y}_j),$$
$$\gamma_1 \widetilde{\Phi}(\boldsymbol{y}_j) = \beta \varphi(\boldsymbol{y}_j).$$

x: pre-selected source points $(\boldsymbol{z}_\ell)$,
•: nodes on the target patch $(\boldsymbol{y}_\ell)$.

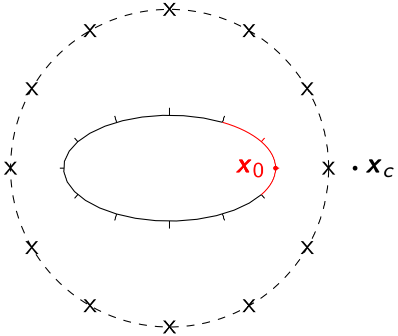
Figure: Construction of local density interpolant using source points.

Problem: rigorous estimate of $\|\varphi - \tilde{\varphi}\|_{L^\infty_\tau}$.

Numeric test

Test identity:

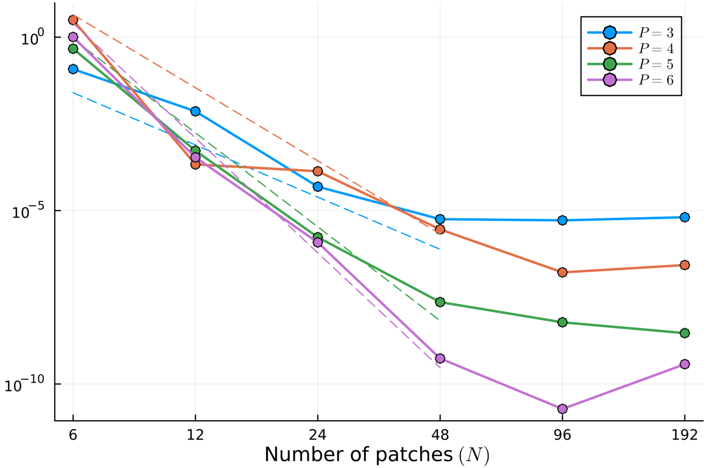
$$\int_{\Gamma} \left(\partial_n G(\boldsymbol{y}, \boldsymbol{x}_0) G(\boldsymbol{y}, \boldsymbol{x}_c) - G(\boldsymbol{y}, \boldsymbol{x}_0) \partial_n G(\boldsymbol{y}, \boldsymbol{x}_c)\right) \mathrm{d}s_{\boldsymbol{y}} = G(\boldsymbol{x}_0, \boldsymbol{x}_c)/2$$



- N: number of patches;
- P: number of Gauss-Legendre nodes on each patch.

Numeric test

Expected error: $O(N^{-(2P-1)} + \varepsilon(P)N^{-1})$



Semi-analytic construction

In the case of 2-d Laplace's equation, (1) becomes

$$\begin{cases} \Delta\Phi = 0, & \text{in } \Sigma; \\ \Phi = \alpha\varphi, & \text{on } \tau; \\ \partial_{\mathbf{n}}\Phi = \beta\varphi, & \text{on } \tau. \end{cases}$$

We search $\Phi = \mathrm{Re}\,(f)$, where f is holomorphic in Σ satisfying

$$\begin{cases} \mathrm{Re}\,(f) = \alpha\varphi & \text{on } \tau; \\ \mathrm{Im}\,(\partial_s f) = \beta\varphi & \text{on } \tau, \end{cases}$$

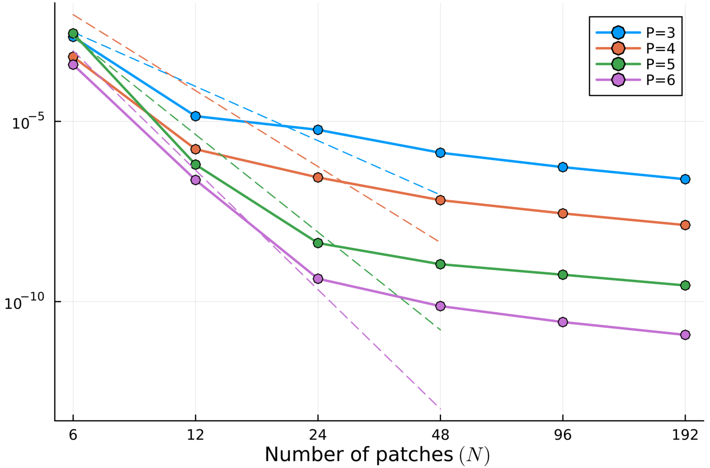
which gives the solution

$$f = \alpha\hat{\varphi} + i\beta\int\hat{\varphi}\,\mathrm{d}s,$$

where $\hat{\varphi}$ is the analytic extension of φ.

Numeric test

Expected error: $O(N^{-(2P-1)} + \varepsilon(P)N^{-1})$



Conclusion

Local density interpolation: correcting singular integrals with Green's identity.

	Source points	Semi-analytic
Error estimate	hard	simple
Generalization to other equations	simple	hard

Table: Comparison of two approaches to constructing the interpolant.