Low to high order finite element resolution for elliptic problems in the presence of a Dirac source term

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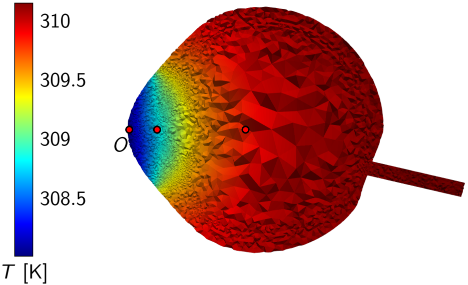
Context

- Previous work[1]: reduced order modeling of heat transfer within the human eyeball, in 3D, focusing on studying the sensitivity of temperature outputs to model parameters.

- Problem of the form $-\nabla \cdot (k \nabla u) = f$, with various boundary conditions, with an output of interest $s = \ell(u) := \langle \delta_O, u \rangle$.
- Dirac possibly on the border of the domain,
- Applications in various domains (accoustic, elasticity, etc.).



[1] Thomas Saigre et al. "Model order reduction and sensitivity analysis for complex heat transfer simulations inside the human eyeball". Minor revision in IJNMBE. Dec. 2023.

Reduced basis method

In the context of certified reduced basis method (RBM), to recover **quadratic convergence** in the output of the reduced solution, we need to compute the solution of the dual problem[1]: find $\psi(\mu) \in H^1(\Omega)$ such that

$$a(u, \psi(\mu); \mu) = -\ell(u) \quad \forall u \in H^1_0(\Omega)$$

Issue: the functionnal $\ell\colon u \mapsto \langle \delta_O, u \rangle$ is not well defined in H^1, so the usual theory (Lax-Milgram, results of minimization) does not apply.

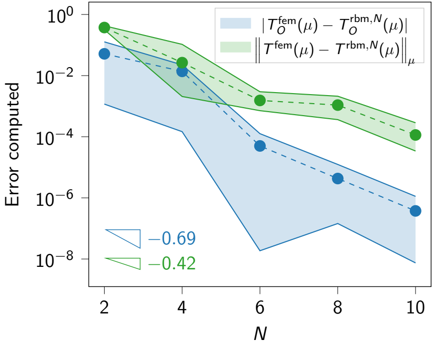


Figure 1: Convergence of the errors on the field and the output on point O, using the dual problem.

[1] Prud'homme et al. "Reliable Real-Time Solution of Parametrized Partial Differential Equations: Reduced-Basis Output Bound Methods". In: *Journal of Fluids Engineering* (2001)

Description of the benchmark problem

Let $\boldsymbol{X}_0 = (x_0, y_0)$ be a point in a domain $\Omega \subset \mathbb{R}^2$. We vary the position of $\boldsymbol{X}_0$.
We consider the following elliptic problem:

$$\begin{cases} -\Delta u = \delta_{\boldsymbol{X}_0} & \text{in } \Omega \\ + \text{ boundary conditions on } \partial\Omega \end{cases}$$

- Dirichlet boundary conditions: $u = 0$ on $\partial\Omega$,
- Neumann boundary conditions: $\partial_{\boldsymbol{n}} u = 0$ on $\partial\Omega$,
- Robin boundary conditions: $\partial_{\boldsymbol{n}} u + \mu u = g$ on $\partial\Omega$,
- Mixed boundary conditions, on Γ_{left} and Γ_{right}.

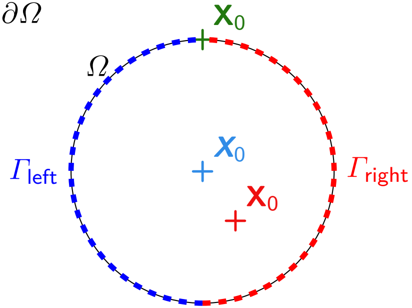


Figure 2: Description of the domain Ω.

Computation of the error

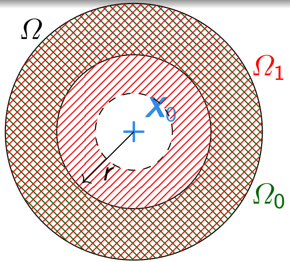
Convergence theorem [Nitsche and Schatz 1974]

Let $\Omega_0 \subsetneq \Omega_1 (\subsetneq \Omega)$, both not containing the singularity. If the solution u_h is regular enough in Ω_1, then the solution is converging at the order k for the norm $\|\cdot\|_{H^1(\Omega_0)}$.

Convergence theorem [Bertoluzza et al. 2018]

The $\mathbb{P}_k$-finite element method converges at the order k on a domain $\Omega_0 \subsetneq \Omega$ not containing the singularity.

- Treated in theory for Dirichet boundary conditions.
- The error is computed on points that are « **far** » from the singularity.
- Domain considered: $\Omega_0 = \Omega \setminus B(\boldsymbol{X}_0, r)$.



State of the art

- The solution of the problem is not regular,
- Even if the problem is not well defined, we can write the discret problem with the finite element method.
- First result[2]: for $p \geqslant 0$,

$$\|u - u_h\|_{H^1(\Omega_0)} \leqslant C(\Omega_0, \Omega_1, \Omega) \left(h^k \|u\|_{H^{k-1}(\Omega_1)} + \|u - u_h\|_{H^{-p}(\Omega_1)} \right)$$

- Improved by Bertoluzza et al. 2018[3]:

$$\|u - u_h\|_{H^1(\Omega_0)} \leqslant C(\Omega_0, \Omega_1, \Omega) h^k \sqrt{|\log(h)|}$$

[2] Nitsche et al. "Interior estimates for Ritz-Galerkin methods" (1974)
[3] Bertoluzza et al., "Local error estimates of the finite element method for an elliptic problem with a Dirac source term", Numerical Methods for Partial Differential Equations (2018)

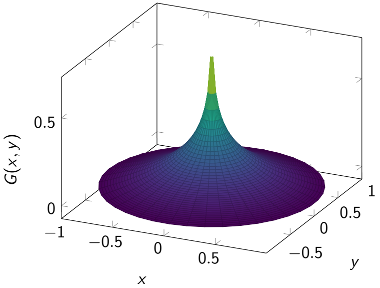
Reference solution

Green function provides the reference solution

The *Green function* $G\colon \Omega \to \mathbb{R}$ defined by

$$G(x,y) = -\frac{1}{2\pi}\log\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)$$

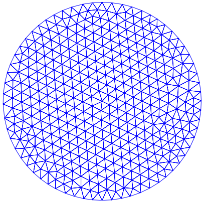
satisfies $-\Delta G = \delta_{\mathbf{x}_0}$ in Ω.



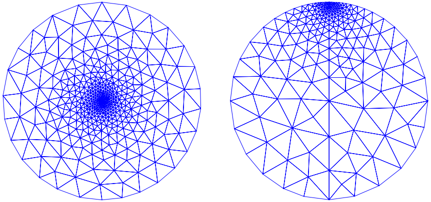
Finite element resolution

- Discretization of the geometry, with a mesh size h,
- The position of the Dirac is **not** associated with a node of the mesh,
- Construction of the **Lagrange finite element space** V_h^k of $\mathcal{C}^0$ polynomials of degree k,
- Implementation of the right-hand side using the `SensorPointwise` of the open-source library Feel++[a].

[a] Feel++ github.com/feelpp/feelpp



(a) Initial mesh.



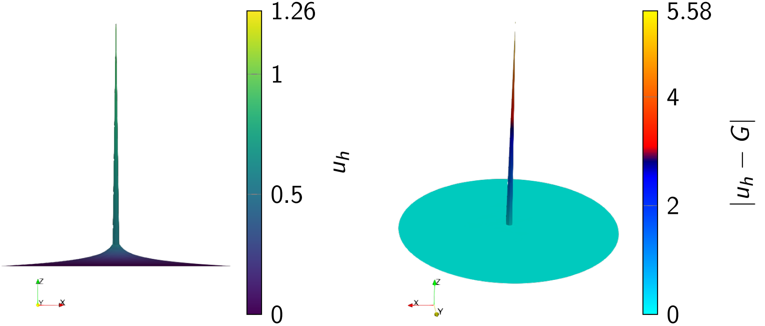
(b) Remeshed around the Dirac

Figure 3: Meshing the geometry.

Example of solution

Order	Ndof
$\mathbb{P}_1$	20,614
$\mathbb{P}_2$	82,400
$\mathbb{P}_3$	$1.85 \cdot 10^5$
$\mathbb{P}_4$	$3.29 \cdot 10^5$
$\mathbb{P}_5$	$5.15 \cdot 10^5$
$\mathbb{P}_6$	$7.41 \cdot 10^5$

Table 1: Number of degrees of freedom, $h = 0.0125$.



(a) Numerical solution u_h.

(b) Error on the numerical solution u_h compared to the exact solution G.

Figure 4: Solution with order $\mathbb{P}_1$, and $h = 0.0125$.

Convergence of the error, for the Dirichet problem[4]

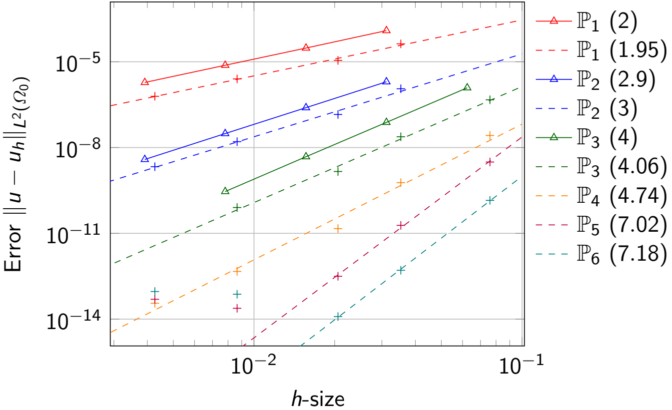


Figure 5: Error $L^2(\Omega_0)$ $(r = 0.2)$. Results from literature in plain lines, present work dashed lines.

[4] Bertoluzza et al., "Local error estimates of the finite element method for an elliptic problem with a Dirac source term", Numerical Methods for Partial Differential Equations (2018)

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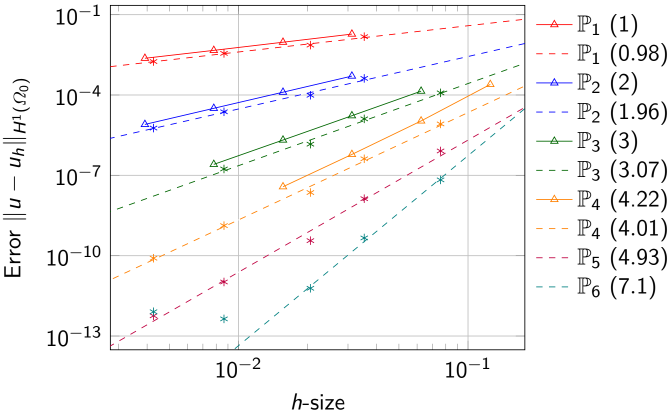


Figure 5: Error $H^1(\Omega_0)$ $(r = 0.2)$. Results from literature in plain lines, present work dashed lines.

[4] Bertoluzza et al., "Local error estimates of the finite element method for an elliptic problem with a Dirac source term", Numerical Methods for Partial Differential Equations (2018)

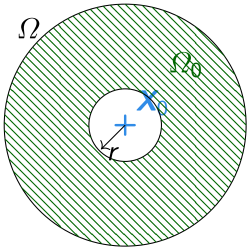
Impact of the size of the radius r

Back to theoretical estimates

- Between Ω_0 and Ω_1, there should be N layers of elements, where N depends on the method, and increases with the order of discretization.
- Heuristically: 1 layer per order of discretization.

Q: Is the theory sharp or pessimistic?

- The mesh size h is fixed (0.0125 and 0.00625)
- We study of the evolution of the error with the radius r.



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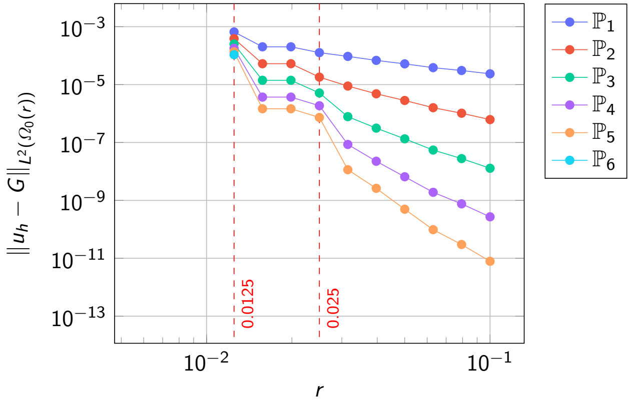
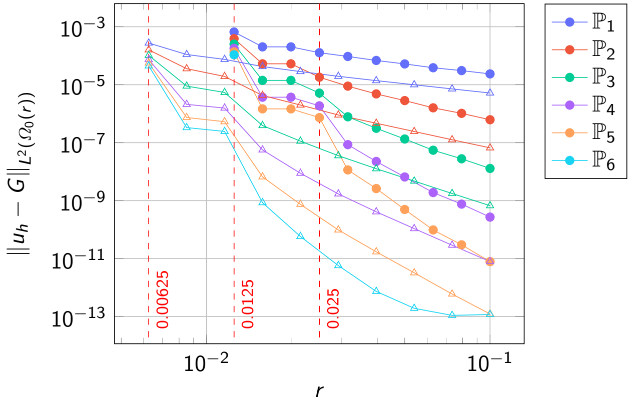


Figure 6: Dot marks: $h = 0.0125$, triangle marks: $h = 0.00625$.

- The error is degraded when $r \leqslant 2h$.
- **Independent** of the order of discretization, up to the constants in the estimate.

Impact of the size of the radius r



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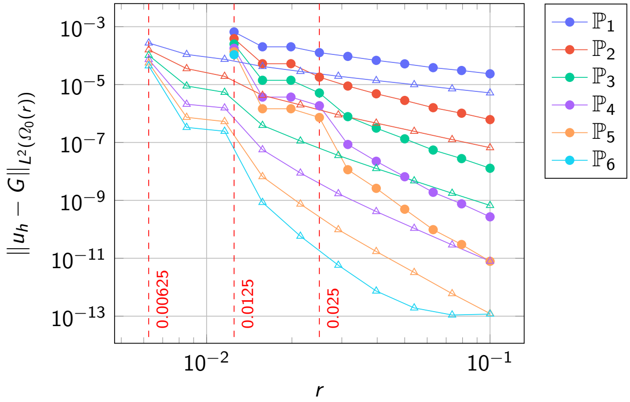

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Q: Is the theory sharp or pessimistic?

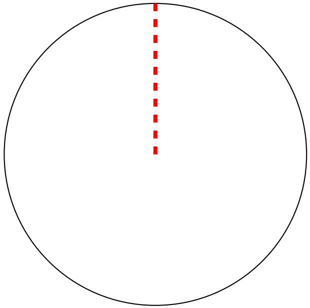
A: Pessimistic

Impact of the singularity position on the error

- The singularity moves on the y-axis, from the center of the domain to the border.
- Error estimate[5]on the border $\partial\Omega$:

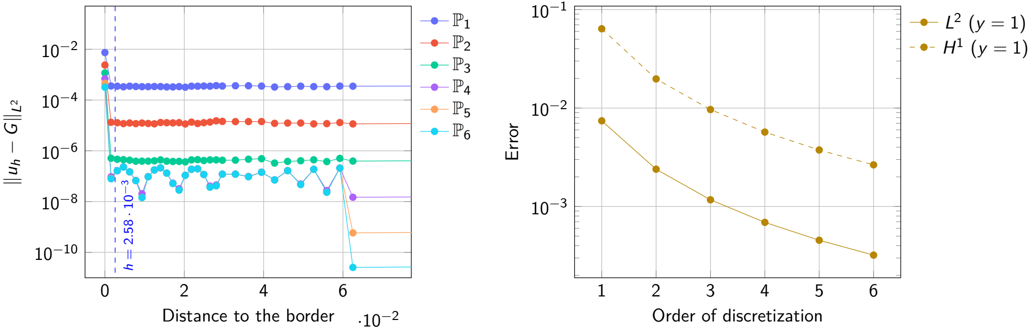
$$\|u - u_h\|_{H^1(\Omega_0)} \leqslant C$$

- **Q:** How to account for Dirichet boudary condition and the presence of the Dirac source term on the boundary at the same time ?



[5] Silvia Bertoluzza et al. "Local error estimates of the finite element method for an elliptic problem with a Dirac source term". In: *Numerical Methods for Partial Differential Equations* 34.1 (2018), pp. 97–120

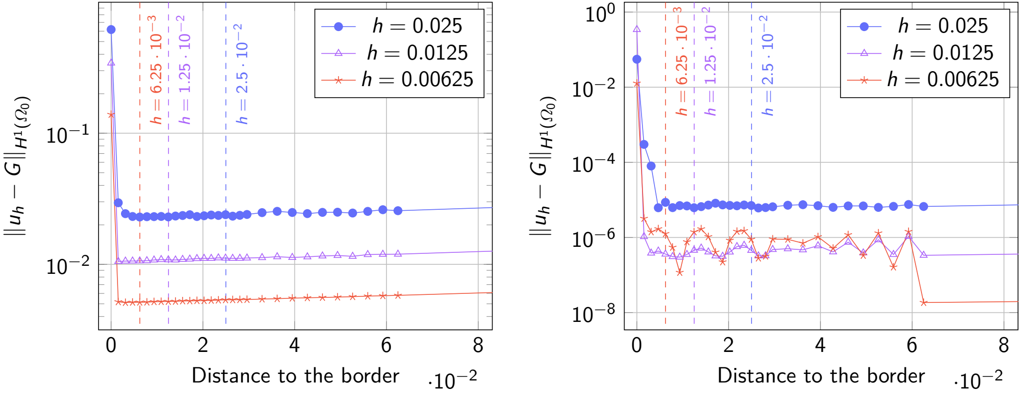
Impact of the singularity position on the error



(a) $d(\boldsymbol{X}_0, \partial\Omega) \in [0, 1]$. (b) Zoom on $d(\boldsymbol{X}_0, \partial\Omega) = 0$.

Figure 7: Evolution of the error L^2 over the domain Ω_0, dependence on the distance of the discontinuity to the border of the domain, for the Robin problem.

Impact of the singularity position on the error



(a) $\mathbb{P}_1$ discretization (b) $\mathbb{P}_4$ discretization

Figure 7: Comparison of the error for various mesh sizes, for the **Robin** problem.

Conclusion and outlook

- Study of the convergence of the finite element method for the resolution of an elliptic problem with a Dirac source term:
  - **same behavior** is observed for other boundary conditions,
  - other results, including geometric singularities, have been studied
- Theory vs. numerical results: impact of r
  - Theoretical convergence estimate are **recovered** in Ω_0,
  - Theory is **too pessimistic** with respect to the definition of Ω_0.
  - r affects the constant in the estimates for with at least two layers of elements, the convergence on h being **still optimal**.

Conclusion and outlook

- Theory vs. numerical results: impact of the position of singularity close to the boundary
  - Theory dictates **only boundedness** on the error,
  - Numerical experiments show that the **error decreases**,
  - Higher order approximations are **better**.
- **Future wokrs:**
  - Refine the theory,
  - get a better understanding of some case studies.

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Thank you for your attention !

References

[Ber+18] Silvia Bertoluzza et al. "Local error estimates of the finite element method for an elliptic problem with a Dirac source term". In: *Numerical Methods for Partial Differential Equations* 34.1 (2018), pp. 97–120.

[NS74] Joachim A. Nitsche and Alfred H. Schatz. "Interior estimates for Ritz-Galerkin methods". In: *Mathematics of Computation* 28.128 (1974), pp. 937–958.

[Pru+01] C. Prud'homme et al. "Reliable Real-Time Solution of Parametrized Partial Differential Equations: Reduced-Basis Output Bound Methods". In: *Journal of Fluids Engineering* 124.1 (Nov. 2001), pp. 70–80.

[Pru+24] Christophe Prud'homme et al. *feelpp/feelpp: Feel++ Release V111 preview.9*. Version v0.111.0-preview.9. Mar. 2024.

[SPS23] Thomas Saigre, Christophe Prud'Homme, and Marcela Szopos. "Model order reduction and sensitivity analysis for complex heat transfer simulations inside the human eyeball". Minor revision in IJNMBE. Dec. 2023.

Example of implementation in FEEL++[5]

```
auto a = form2( _trial = Vh, _test = Vh);
a = integrate( _range = elements(mesh), _expr = gradt(u) *
↪  trans(grad(v)) );
a += on(_range = boundaryfaces(mesh), _rhs = l, _element = u,
↪  _expr = G_expr);

auto s = std::make_shared<SensorPointwise<space_type>>(Vh, n,
↪  "0");
auto l = form1( _test = Vh, _vector = s->containerPtr() );
a.solve(_rhs=l, _solution=u);
```

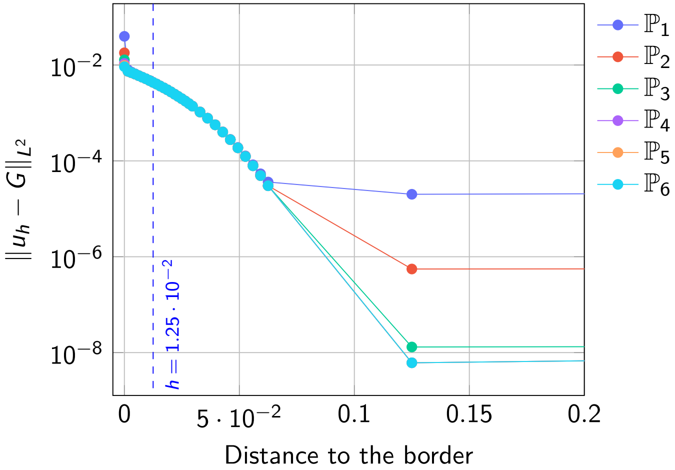
[5] github.com/feelpp/feelpp

Regularized problem

We consider the regularized problem:

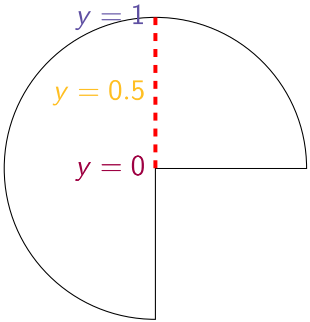
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $f(\boldsymbol{X}) = \exp\left(\frac{X - \boldsymbol{X}_0}{2r^2}\right)$.
We take $r = 2h$.



Impact on the error convergence

- We study the impact of the position of the singularity on the error convergence rate.
- Another domain considered.



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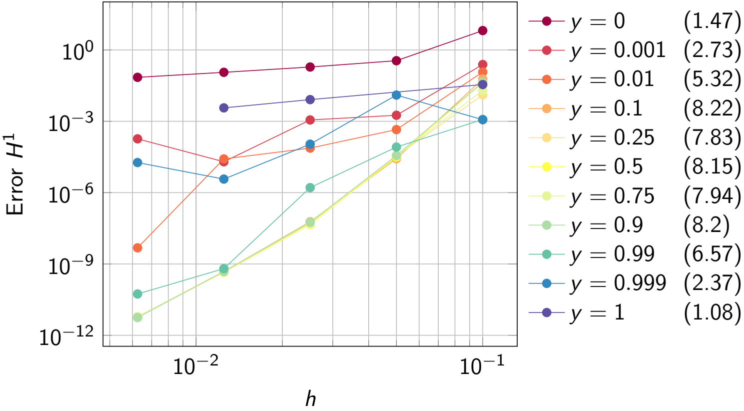

Figure 8: Convergence curve with $\mathbb{P}_6$ discretization for Dirac position going from the internal corner $(y = 0)$ to the border of the domain $(y = 1)$