Singular perturbation method for the stability analysis of coupled systems

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Systems with different time-scales

- Many natural phenomena feature interaction of processes on different times scales. Electro-mechanical systems are one of the several examples of interest.
- A lot of difficulties appear. For instance, huge cost in numerical simulations since the fastest time scale sub-system must be fully solved over a timespan of the slowest scales' order.
- **Desirable** : we want instead solve a limit system, describing approximately the full behavior when some parameters (representing the scales) go to zero (or infinity)

General Question

Consider general linear finite-dimensional systems :

$$\begin{cases} \varepsilon\dot{y} = A_1 y + B_1 C_1 z, & \textbf{fast dynamics} \\ \dot{z} = A_2 z + B_2 C_2 y, & \textbf{slow dynamics} \\ y(0) = y_0, \ z(0) = z_0 \end{cases}$$

with $y \in \mathbb{R}^n$, $z \in \mathbb{R}^m$ and the matrices A_i, B_i, C_i of appropriate dimension. One looks for conditions on the involved matrices such that, setting $H = \mathbb{R}^n \times \mathbb{R}^m$

$$\|(y,z)\|_H \leq Ce^{-\mu t}\|(y_0,z_0)\|_H$$

Question

Supposing ε small enough, is it possible to obtain "easy" stability conditions ?

Answer : apply the singular perturbation method !

The singular perturbation principle

The singular perturbation principle consists in decoupling the coupled system into two approximated systems :

1. The **reduced order system** $\simeq$ **slower system**
2. The **boundary layer system** $\simeq$ **faster system**

Question

How can one compute these two systems ?

Reduced order system

Suppose that $\varepsilon = 0$ and that A_1 is invertible, then :

$$y = -A_1^{-1} B_1 C_1 z.$$

Plugging this equality in the second equation, we obtain

Reduced order system

$$\dot{\bar{z}} = (A_2 - B_2 C_2 A_1^{-1} B_1 C_1)\bar{z}$$

Boundary layer system

Set $\tau = \frac{t}{\varepsilon}$ and $\tilde{y} = y + A_1^{-1}B_1C_1z$. Then,

$$\frac{d}{d\tau}\bar{y} = A_1 \underbrace{\left(y + A_1^{-1}B_1C_1z\right)}_{=\bar{y}} + \varepsilon \frac{d}{dt}A_1^{-1}B_1C_1z$$

Taking $\varepsilon = 0$, one obtains

Boundary layer system

$$\frac{d}{d\tau}\bar{y} = A_1\bar{y}.$$

Result in finite dimension

Result

There exists $\varepsilon^* > 0$ such that, for every $\varepsilon \in (0, \varepsilon^*)$:

Boundary layer system stable + Reduced order system stable $\Rightarrow$ Full-system stable.

Such a result can be found for instance in [Kokotović, Khalil, O'Reilly, 1986]. The strategy relies on a frequency approach.

Question

What about the infinite-dimensional case ?

An interesting counter-example

Very few results exist for the infinite-dimensional setting : [Tang, Mazanti, 2017], [Cerpa, Prieur, 2020]. These results focus on **hyperbolic systems**

An interesting counter-example

Counter-example

$$\begin{cases} \varepsilon \dot{y}(t) = -0.1y(t) - z(1,t) \\ z_t(t,x) + z_x(t,x) = 0 \\ z(0,t) = 2z(1,t) + 0.2y(t). \end{cases}$$

The reduced order system is given by

$$\begin{cases} \bar{z}_t(t,x) + \bar{z}_x(t,x) = 0 \\ \bar{z}(0,t) = 0 \end{cases}$$

and the boundary layer system reads

$$\frac{d}{d\tau}\bar{y}(\tau) = -0.1\bar{y}(\tau).$$

Both systems are **always exponentially stable**. But the full-system is not (proof based on the **method of characteristics**).

An interesting counter-example

Question

Is it still possible to obtain general results ?

Answer : yes, but under restrictive assumptions.

Abstract systems

The semigroup framework is adopted with H_i and U_i Hilbert spaces ($i = 1$ or 2).

Abstract systems

$$\begin{cases} \varepsilon \dot{y} = \mathcal{A}_1 y + \mathcal{B}_1 \mathcal{C}_1 z, \\ \dot{z} = \mathcal{A}_2 z + \mathcal{B}_2 \mathcal{C}_2 y, \\ y(0) = y_0, \ z(0) = z_0 \end{cases}$$

1. The operators $\mathcal{A}_i : D(A_i) \subset H_i \to H_i$ generate strongly continuous semigroups $(\mathbb{T}_i(t))_{t\geq 0}$;
2. $B_i \in \mathcal{L}(U_i, D(\mathcal{A}_i^*)')$ admissible for $(\mathbb{T}_i(t))_{t\geq 0}$;
3. $C_1 \in \mathcal{L}(D(\mathcal{A}_2, U_2))$ (resp. $C_2 \in \mathcal{L}(D(\mathcal{A}_1, U_2))$) admissibles.

Remarks

The operators $\mathcal{A}_i$ can represent partial derivatives.

Additional assumptions

The counter-example dealt with a fast ODE coupled with a slow PDE.
General results can be obtained under this assumption :

Finite-dimension assumption

The Hilbert spaces H_2 and U_2 are of finite-dimensions.

Additional assumptions

Moreover, one needs to assume that

Stability assumptions

The semigroups generated by $\mathcal{A}_1$ and $\mathcal{A}_2 - \mathcal{B}_2\mathcal{C}_2\mathcal{A}_1^{-1}\mathcal{B}_1\mathcal{C}_1$ are exponentially stable.

Results on abstract systems

Theorem (M., 2024)

Under the latter assumptions, there exists ε^* such that, for every $\varepsilon \in (0, \varepsilon^*)$, the full-system is exponentially stable.

Unlike the finite-dimensional case, the proof of the latter result is based on the construction of Lyapunov functional.

Tikhonov's theorem (M., 2024)

Suppose that

$$\|y_0\|_{H_1} = O(\varepsilon), \ \|z_0\|_{H_2} = O(\varepsilon),$$

then, there exists ε^* such that, for every $\varepsilon \in (0, \varepsilon^*)$, we have $\|y(t)\|_{H_1} = O(\varepsilon)$ and $\|z(t)\|_{H_2} = O(\varepsilon)$ for all $t \geq 0$.

Sketch of the proof : ISS property

Since the semigroup generated by $\mathcal{A}_1$ is **exponentially stable**, there exists a symmetric and positive definite operator $\mathcal{P} \in \mathcal{L}(H_1)$ such that, for every

$$\langle P\mathcal{A}_1 y, y\rangle_{H_1} + \langle Py, \mathcal{A}_1 y\rangle_{H_1} = -\|y\|_{H_1}^2$$

Input-to-state stability (ISS) property

For every $d \in U_1$, the derivative of the **Lyapunov functional** $V = \langle Py, y\rangle_{H_1}$ along the trajectories of

$$\frac{d}{dt}y = \mathcal{A}_1 y + \mathcal{B}_1 d,$$

satisfies

$$\frac{d}{dt}V(y) \leq -\alpha\|y\|_{H_1}^2 + \beta\|d\|_{U_1}^2, \ \alpha, \beta > 0.$$

Sketch of the proof : Forwarding approach

Forwarding approach (Mazenc, Praly, 1996)

Based on the introduction of a Lyapunov functional

$$W(y,z) = \varepsilon V(y) + \|z - My\|_{H_2}^2,$$

with M defined as $M = \mathcal{B}_2\mathcal{C}_2\mathcal{A}_1^{-1}$ which is **bounded** (due to the boundedness assumption).

Proof

ISS property + Forwarding Lyapunov functional yield the desired result.

Explaining the counter-example

Question

What happens in the counter example ?

Lyapunov functionals cannot provide counter-examples in general. One needs rather to study the **spectrum**.

Tranport Equation/ODE

System

$$\begin{cases} \varepsilon \dot{y} = Ay + Bz(t,1) \\ z_t + \Lambda z_x = 0 \\ z(t,0) = G_1 z(t,1) + G_2 y(t) \\ y(0) = y_0, \ z(0,x) = z_0(x). \end{cases}$$

$y(t) \in \mathbb{R}^n, \ z(t,\cdot) \in L^p(0,1;\mathbb{R}^m)$ and the involved matrices are of appropriate dimension.

Reduced order system

$$\begin{cases} \bar{z}_t + \Lambda \bar{z}_x = 0 \\ \bar{z}(t,0) = (G_1 - G_2 A^{-1} B)\bar{z}(t,1) \\ \bar{z}(0,x) = \bar{z}_0(x) \end{cases}$$

Tranport Equation/ODE

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$y(t) \in \mathbb{R}^n,\ z(t,\cdot) \in L^p(0,1;\mathbb{R}^m)$ and the involved matrices are of appropriate dimension.

Boundary layer system

$$\frac{d}{d\tau}\bar{y}(t) = A\bar{y}(t),\ \tau := \frac{t}{\varepsilon}$$

The uncoupled PDE

We already know that

Boundary layer system & Reduced order system stable $\not\Rightarrow$ full-system stable.

We have even more :

Proposition (Arías, M., Mazanti, 2024)

If the **uncoupled system** is unstable, meaning that

$$\begin{cases} z_t + \Lambda z_x = 0, \\ z(t,0) = G_1 z(t,1), \end{cases}$$

is unstable, then the full-system is unstable for all $\varepsilon > 0$

Going back to the counter-example

Counter-example

$$\begin{cases} \varepsilon \dot{y}(t) = -0.1y(t) - z(1,t) \\ z_t(t,x) + z_x(t,x) = 0 \\ z(t,0) = 2z(t,1) + 0.2y(t). \end{cases}$$

Uncoupled system

$$\begin{cases} z_t + z_x = 0, \\ z(t,0) = 2z(t,1), \\ z(0,x) = z_0(x). \end{cases}$$

It is clearly **unstable**.

An approximation result

Theorem (Arías, M., Mazanti, 2024)

Suppose that

- The **uncoupled system** and the **boundary layer system** are stable.
- The **reduced order system** satisfies the **Hale-Silkowski criterion**,

then there exists ε^* such that, for all $\varepsilon \in (0, \varepsilon^*)$, the full-system is stable.

Reduced order system

$$\begin{cases} \bar{z}_t + \Lambda \bar{z}_x = 0 \\ \bar{z}(t,0) = (G_1 - G_2 A^{-1} B)\bar{z}(t,1) \\ \bar{z}(0,x) = \bar{z}_0(x) \end{cases}$$

An approximation result

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Hale-Silkowski criterion

The uncoupled system satisfies this criterion if

$$\max\{\rho\left((G_1 + G_2(i\eta \mathrm{I}_n - A)^{-1}B)\mathrm{diag}\{\delta_1 e^{i\theta_1}, \dots, \delta_m e^{i\theta_m}\}\right), \\ \theta_k \in \mathbb{R}, \delta_k \in [0,1], \forall k \in \{1, \dots, m\}\} < 1,$$

for any $\eta \geq 0$.

Achievements and open problems

Achievements

1. Some extensions of the the singular perturbation method have been found ;
2. **Counter-examples** have been obtained (and explained)

Open problems

1. What about the case of **coupled PDEs** ?
2. Is this possible to obtain simpler **controllability criteria** for multi-scale systems ?

Thank you for your attention

Any question ?