On a Neural Implementation of Brenier's Polar Factorization

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In this talk, we delve into Brenier's theorem [1] that extends the polar decomposition from square matrices to vector field. Known as the polar factorization theorem, it asserts that any field $F : \mathbb{R}^d \to \mathbb{R}^d$ can be recovered as the composition of the gradient of a convex function u with a measure-preserving map M, namely $F = \nabla u \circ M$. In [2], we propose a practical implementation of this factorization and explore possible uses within machine learning. The theorem is closely related to optimal transport (OT) theory, and we borrow from recent advances in the field of neural optimal transport to parameterize the potential u as an input convex neural network. The map M can be either evaluated pointwise using u^*, the convex conjugate of u, through the identity $M = \nabla u^* \circ F$, or learned as an auxiliary network. Because M is, in general, not injective, we consider the additional task of estimating the ill-posed inverse map that can approximate the pre-image measure M^{-1} using a stochastic generator. We illustrate possible applications of Brenier's polar factorization to non-convex optimization problems, as well as sampling of densities that are not log-concave.

[1] Y. Brenier. *Polar factorization and monotone rearrangement of vector-valued functions.* Communications on pure and applied mathematics, **44(4)**, 375–417, 1991.

[2] N. Vesseron, M. Cuturi. *On a neural implementation of brenier's polar factorization*, 2024.

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