Coupling kinetic and fluid equations: Linear (in)stability in fluid particles interactions

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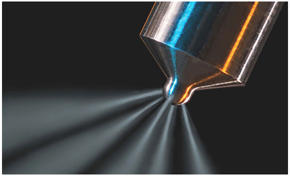
Sprays

Dispersed phase of particles (droplets, dust specks) within a gas



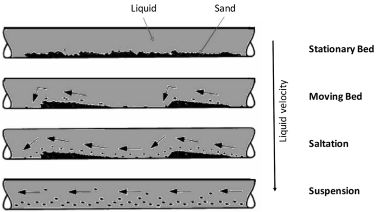
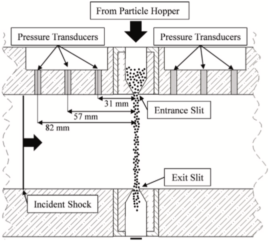


Fig. 1. Sand flow regime in horizontal pipelines.

(c) Leporini at al (2019)



(d) Daniel-Wagner (2022)

Modeling of a spray: fluid-kinetic coupling

- Unknowns for the gas: macroscopic quantities $\to$ fluid mechanics equation

$$\varrho(t,\boldsymbol{x}) \geq 0, \quad \boldsymbol{u}(t,\boldsymbol{x}) \in \mathbf{R}^3, \quad e(t,\boldsymbol{x}) \geq 0, \quad \alpha(t,\boldsymbol{x}) \in (0,1].$$

- Unknown for the dispersed phase : mesoscopic description $\to$ kinetic distribution function

$$f(t,\boldsymbol{x},\boldsymbol{v}) \geq 0$$

with $\boldsymbol{v}$ the velocity of the droplets.

Kinetic moments

$$\varrho_p(t,x) = \int f(t,x,v)\,\mathrm{d}v, \quad j_p(t,x) = \int vf(t,x,v)\,\mathrm{d}v$$

Hypothesis. The particles are monodisperse: all particle have the same radius $r_p > 0$.

Critical quantity: fluid volume fraction

$$\alpha(t,\mathbf{x}) = 1 - \frac{4}{3}\pi r_p^3 \int f(t,\mathbf{x},\mathbf{v})\,\mathrm{d}v$$

A first example: Thin sprays $\alpha(t,x) \approx 1$

Euler equations coupled with a Vlasov equation through a friction force

$$\begin{cases} \partial_t f + \mathbf{v}\cdot\nabla_x f + D_\star \nabla_v \cdot ((\mathbf{u}-\mathbf{v})f) = 0 \\ \partial_t \varrho + \nabla_x \cdot (\varrho \mathbf{u}) = 0 \\ \partial_t(\varrho\mathbf{u}) + \nabla_x\cdot(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho) = D_\star \int_{\mathbb{R}^3} (\mathbf{v}-\mathbf{u}) f \,\mathrm{d}v. \end{cases}$$

with pressure $p = \varrho^\gamma$.

The model problem: thick sprays $\alpha(t,x) \in (0,1]$

Compressible Vlasov-Euler equation. Coupling through a friction force and volume fraction.

$$\begin{cases} \partial_t f + \mathbf{v}\cdot\nabla_x f - \nabla_x p(\varrho)\cdot\nabla_v f + D_\star \nabla_v \cdot ((\mathbf{u}-\mathbf{v})f) = 0 \\ \partial_t(\alpha\varrho) + \nabla_x\cdot(\alpha\varrho\mathbf{u}) = 0 \\ \partial_t(\alpha\varrho\mathbf{u}) + \nabla_x\cdot(\alpha\varrho\mathbf{u}\otimes\mathbf{u}) + \alpha\nabla_x p(\varrho) = D_\star \int_{\mathbb{R}^3}(\mathbf{v}-\mathbf{u})f\,\mathrm{d}v. \end{cases}$$

With the fluid volume fraction $\alpha = 1 - \frac{4}{3}\pi r_\star^3 \int_{\mathbf{R}^3} f\,\mathrm{d}v$.
Our interest is in all regime $0 < \alpha \leq 1$.
This model is already very simplified compared to actual engineer model (See Fox[23]).

L. Boudin, L. Desvillettes, and R. Motte. A modeling of compressible droplets in a fluid. Commun. Math. Sci., 1(4):657-669, 2003.

Some bibliography

- Models
  - Models explicitly written as PDEs Williams [74'], Classification of sprays O'Rourke [81'].
  - Discretization in Dukowicz [80'], advanced methods Fox [23'].
- Mathematical theory of Thin Sprays ($\alpha \approx 1$):
  - Local-in-time well posedness for strong solution Baranger-Desvillettes [06'], Mathiaud [10']
  - Global existence of weak solution in various settings, Anoshchenko-Boutet de Monvel-Berthier[97'], Boudin-Desvillettes-Grandmont-Moussa[09'] and others
  - Large time behavior studied Han Kwan-Moussa-Moyano[20'], Ertzbischoff-Han Kwan[21']
- Mathematical theory of Thick Sprays ($0 < \alpha \leq 1$) :
  - Work on models Boudin-Desvillettes-Mottes [03']
  - Numerical work by Benjelloun-Desvillettes-Ghidaglia-Nielsen [12']
  - Linear stability studied in Buet-Després-Desvillettes [22']
  - Local Well-posedness for a regularized thick sprays model Buet-Després-F [23']
  - Local in time Well-posedness for the Navier Stokes (with diffusion) case and Penrose stable initial data Ertzbischoff-Han Kwan [23']

Linearized thick sprays model

For convenience, assume that there is no friction $D_\star = 0$

- Start from nonlinear thick sprays equations with no friction $D_\star = 0$

$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_x f - \nabla_x p(\varrho) \cdot \nabla_v f = 0, \quad \mathbf{x} \in \mathbb{T}^3, \quad \mathbf{v} \in \mathbb{R}^3, \\ \partial_t(\alpha\varrho) + \nabla_x \cdot (\alpha\varrho\mathbf{u}) = 0, \quad \mathbf{x} \in \mathbb{T}^3, \\ \partial_t(\alpha\varrho\mathbf{u}) + \nabla_x \cdot (\alpha\varrho\mathbf{u} \otimes \mathbf{u}) + \alpha\nabla_x p(\varrho) = 0, \quad \mathbf{x} \in \mathbb{T}^3. \end{cases}$$

- Linearize around Gaussian in v for the particles

$$\begin{cases} \varrho_0(x) = cte \\ u_0(x) = 0 \\ f_0(v) \text{ given} \\ \\ \varrho(t,x) = \varrho_0 + \varepsilon\varrho_1(t,x) + O(\varepsilon^2) \\ u(t,x) = 0 + \varepsilon u_1(t,x) + O(\varepsilon^2) \\ f(t,x,v) = f_0(v) + \varepsilon\sqrt{f_0(v)} f_1(t,x,v) + O(\varepsilon^2) \end{cases}$$

- It yields the linearized thick sprays equations with $\tau_1 = -\varrho_1/\varrho_0^2$, $c_0^2 = p'(\varrho_0)$, $\alpha_0 = 1 - \frac{4}{3}\pi r_p^3 \int f_0 \,\mathrm{d}v$.

$$\begin{cases} \partial_t f_1 + \mathbf{v} \cdot \nabla_x f_1 - \varrho_0^2 c_0^2 \dfrac{\nabla_v f_0(v)}{\sqrt{f_0(v)}} \cdot \nabla_x \tau_1 = 0. \\ \alpha_0\varrho_0\partial_t\tau_1 = \alpha_0\nabla_x \cdot \mathbf{u}_1 + \dfrac{4}{3}\pi r_p^3 \nabla_x \cdot \displaystyle\int_{\mathbb{R}^3} \mathbf{v}\sqrt{f_0} f_1 \,\mathrm{d}v \\ \alpha_0\varrho_0\partial_t\mathbf{u}_1 = \alpha_0\varrho_0 c_0^2 \nabla_x \tau_1 \end{cases}$$

Finding instabilities

Question: What are the particles profiles $f_0(v)$ which lead to stable or unstable solutions ?
"Pure modes" are solution of the form

$$e(t,x,v,k) = e^{i(kx-\omega(k)kt)}a(v,k), \quad k \in \mathbb{Z},\ \omega(k) \in \mathbb{C}.$$

Can generate classical solution with Fourier series

$$f(t,x,v) = \sum_{k\in\mathbb{Z}} e^{i(kx-\omega(k)kt)}a(v,k)$$

Instabilities II

Take the linearized equations

$$\begin{cases} \partial_t f_1 + v\partial_x f_1 - \dfrac{f_0'(v)}{\sqrt{f_0(v)}}\partial_x \tau_1 = 0. \\ \partial_t \tau_1 = \partial_x u_1 + \partial_x \cdot \displaystyle\int_{\mathbb{R}^3} v\sqrt{f_0} f_1 \,\mathrm{d}v \\ \partial_t u_1 = \partial_x \tau_1 \end{cases}$$

Injecting pures modes solution into this system leads to

$$\begin{cases} (-i\omega + ikv)\alpha(v) = -i\beta \dfrac{kf_0'(v)}{\sqrt{f_0(v)}} \\ -i\omega\beta = ik\gamma + i\displaystyle\int kv\sqrt{f_0(v)}\alpha(v)\,\mathrm{d}v \\ -i\omega\gamma = ik\beta \end{cases}$$

Which lead to the **dispersion relation**

$$\frac{k^2}{\omega^2} + \int \frac{f_0'(v)}{v - \omega/k}\,\mathrm{d}v = 1. \tag{1}$$

Instabilities

Theorem 1: Characterisation of unstable particles profiles

The linearized equations have exponentially growing modes if and only if there exists $k \in \mathbb{Z}$ and $\omega \in \mathbb{C}$ with $\Im m(\omega) > 0$ satisfying

$$\frac{k^2}{\omega^2} + \int \frac{f_0'(v)}{v - \omega/k} = 1.$$

.

Example : two-bumps profile $f_0(v) = e^{-(v-v_0)^2} + e^{-(v+v_0)^2}$, v_0 large enough
Two-stream instability

Stability near a maxwellian profile

Take $f_0(v) = e^{-v^2/2}$. Start from the linearized equations:

$$\begin{cases} \partial_t \tau = \nabla_x \cdot \mathbf{u} + \nabla_x \cdot \int_{\mathbb{R}^3} \mathbf{v}\sqrt{f_0} f \,\mathrm{d}v \\ \partial_t \mathbf{u} = \nabla_x \tau \\ \partial_t f + \mathbf{v} \cdot \nabla_x f - \sqrt{f_0(v)}\mathbf{v} \cdot \nabla_x \tau = 0. \end{cases}$$

The linearized equations in $X = L^2_0(\mathbb{T}^3_x) \times \left(L^2_0(\mathbb{T}^3_x)\right)^3 \times L^2_0(\mathbb{T}^3_x \times \mathbb{R}^3_v)$ rewrites

$$\mathbf{U}'(t) = iH\mathbf{U}(t), \quad H^* = H$$

where the self-adjoint differential operator H is

$$iH = \begin{pmatrix} 0 & \partial_x & \partial_x \int_{\mathbb{R}} v\sqrt{f_0(v)} \cdot \,\mathrm{d}v \\ \partial_x & 0 & 0 \\ v\sqrt{f_0(v)}\partial_x & 0 & -v\partial_x \end{pmatrix}, \quad D\left[H\right] = \left\{U \in X,\ HU \in X\right\}.$$

One has linear stability because H is symmetric

$$\frac{d}{dt}\|\mathbf{U}(t)\|_X = 0$$

Structure of the spectrum

The long time behaviour of $\mathbf{U}(t) := (\tau(t), \mathbf{u}(t), f(t)) = e^{itH}(\tau_{\mathrm{ini}}, u_{\mathrm{ini}}, f_{\mathrm{ini}})$ is characterized by the structure of the spectrum of H.
A self adjoint operator induces the decomposition of the space

$$X = X^{\mathrm{ac}} \oplus X^{\mathrm{sc}} \oplus X^{\mathrm{pp}}.$$

Theorem 2

If $f_0(v) = e^{-v^2/2}$, then H restricted to $X = L^2_0(\mathbb{T}^3_x) \times \left(L^2_0(\mathbb{T}^3_x)\right)^3 \times L^2_0(\mathbb{T}^3_x \times \mathbb{R}^3_v)$ is self adjoint and

- One has the inequality

$$\left\| (H - \lambda - i\varepsilon)^{-1} \mathbf{U} \right\| \leq \frac{C_{\lambda,\mathbf{U}}}{\sqrt{\varepsilon}}, \quad \lambda \in \mathbb{R}, \ \varepsilon > 0.$$

  which yields the decomposition $X = X^{\mathrm{ac}}$.

- As a consequence one obtains a linear damping result

$$\tau(t) \to 0 \text{ and } \mathbf{u}(t) \to 0 \text{ in } L^2_x \text{ strongly}, \ f(t) \rightharpoonup 0 \text{ in } L^2_{x,v} \text{ weakly}.$$

Christophe Buet, Bruno Després, VF. Analog of Linear Landau Damping in a coupled Vlasov-Euler system for thick sprays. 2023. hal-04265990, submitted

Numerical methods

- $$\begin{cases} \partial_t(\alpha\varrho) + \partial_x(\alpha\varrho u) = 0 \\ \partial_t(\alpha\varrho u) + \partial_x(\alpha\varrho u^2) + \alpha\partial_x p = 0 \\ \partial_t f + v\partial_x f - \partial_x p \partial_v f = 0 \\ p(\varrho) = \varrho^\gamma, \gamma = 1.4. \end{cases} \tag{2}$$

- To get rid of the non convervation product $\alpha\partial_x p$, we write the fluid part in conservation form

$$\begin{cases} \partial_t(\alpha\varrho) + \partial_x(\alpha\varrho u) = 0 \\ \partial_t\left(\alpha\varrho u + \frac{4}{3}\pi r_p^3 \int_{\mathbb{R}} vf \,\mathrm{d}v\right) + \partial_x(\alpha\varrho u^2) + \partial_x p + \partial_x\left(\frac{4}{3}\pi r_p^3 \int_{\mathbb{R}} v^2 f \,\mathrm{d}v\right) = 0 \end{cases} \tag{3}$$

- The fluid part writes as $\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}, f) = 0$. The jacobian of $\mathbf{F}$ w.r.t to $\mathbf{U} = (U_1, U_2)$ is

$$\mathbf{A}(\mathbf{U}, f) := \nabla_{\mathbf{U}}\mathbf{F}(\mathbf{U}, f) = \begin{pmatrix} 0 & 1 \\ \frac{-(U_2 - \frac{4}{3}\pi r_p^3 \int_{\mathbb{R}} fv \,\mathrm{d}v)^2}{U_1^2} + \frac{\gamma U_1^{\gamma-1}}{\alpha^\gamma} & \frac{2(U_2 - \frac{4}{3}\pi r_p^3 \int_{\mathbb{R}} fv \,\mathrm{d}v)}{U_1} \end{pmatrix}.$$

The eigenvalues of the matrix $\mathbf{A}(\mathbf{U}, f)$ are

$$\lambda_\pm = u \pm \sqrt{\frac{p'(\varrho)}{\alpha}}.$$

Numerical methods

Given $(\mathbf{U}^n, f^n)$ at a given time t^n.

- Compute f^* by solving the free transport $\partial_t f + v\partial_x f = 0$ with a semi-lagrangian scheme during a timestep Δt with initial condition f^n.
- Compute f^{n+1} by solving $\partial_t f - \partial_x p \partial_v f = 0$ with a semi-lagrangian scheme during a timestep Δt with initial condition f^*.
- Compute $\mathbf{U}^{n+1}$ by solving $\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}, f)$ with a Lax-Wendroff scheme during a timestep Δt with initial condition $(\mathbf{U}^n, f^{n+1})$.
- Given initial data $(\mathbf{U}^n, f^{n+1})$, the quantity $\mathbf{U}^{n+1}$ is computed by the Lax-Wendroff scheme:

$$\mathbf{U}^{n+1} = \mathbf{U}^n - \frac{\Delta t}{2\Delta x}(\mathbf{F}(\mathbf{U}^n_{j+1}, f^{n+1}_{j+1}) - \mathbf{F}(\mathbf{U}^n_{j-1}, f^{n+1}_{j-1}))$$

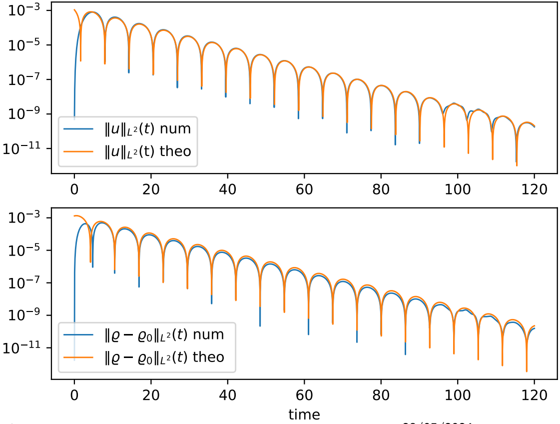
$$+ \frac{\Delta t^2}{2\Delta x^2}\left(\mathbf{A}^n_{j+1/2}(\mathbf{F}(\mathbf{U}^n_{j+1}, f^{n+1}_{j+1}) - \mathbf{F}(\mathbf{U}^n_j, f^{n+1}_j)) - \mathbf{A}^n_{j-1/2}(\mathbf{F}(\mathbf{U}^n_j, f^{n+1}_j) - \mathbf{F}(\mathbf{U}^n_{j-1}, f^{n+1}_{j-1}))\right).$$

Nonlinear simulations: Damping of the acoustic energy

Initial conditions:

$\varrho_0 = 1, \quad u_0 = 0, \quad f_0(x,v) = (1+\varepsilon\cos(kx))e^{-v^2/2}, \quad \varepsilon = 10^{-3}$

Orange curve $\propto e^{\Im m(\omega)t}\cos(\Re e(\omega)t)$, $\omega(k)$ solution of $\frac{k^2}{\omega^2} + \int \frac{f_0'(v)}{v-\omega/k} = 1$



Landau damping in plasma physics

Observation: The linear damping observed in thick sprays has similarities with the Landau damping for Vlasov-Poisson

- First predicted by Landau[46'] for the linearized Vlasov-Poisson system

$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_x f - \mathbf{E} \cdot \nabla_v f_0 = 0, \\ \nabla_x \cdot \mathbf{E} = - \int f \, \mathrm{d}v \end{cases}$$

  around maxwellian equilibrium $f_0(v) = e^{-v^2/2}$

- Landau showed the damping of the electric field

$$\|\mathbf{E}(t)\| = \mathcal{O}\left(e^{\mathrm{Im}(\omega)t} \cos(\mathrm{Re}(\omega))\right),$$

  with $\omega(k) \in \mathbb{C}$ verifies a dispersion relation

$$\int_{\mathbb{R}} \frac{\partial_v f_0(v)}{v - \omega/k} \, \mathrm{d}v = k^2.$$

- To show this, take the ansatz $f(t,x,v) = \alpha(v) e^{-i\omega t} e^{ikx}, \quad E(t,x) = \beta e^{-i\omega t} e^{ikx}$.

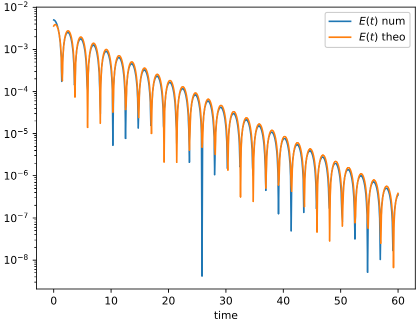


Figure 1: Landau damping for (nonlinear) Vlasov-Poisson

Damping of the acoustic energy with friction

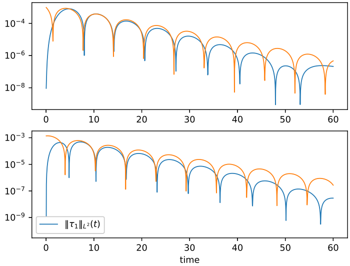


Figure 2: $D_\star = 10^{-3}$

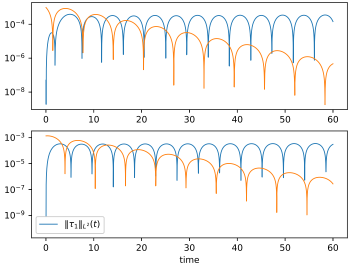


Figure 3: $D_\star = 10^{-1}$

Two-stream instability without friction

Initial conditions:

$$\varrho_0 = 1, \quad u_0 = 0, \quad f_0(x,v) = (1+\varepsilon\cos(kx))(e^{-(v-v_0)^2} + e^{-(v+v_0)^2}), \quad \varepsilon = 10^{-3}$$