Lagrange-Galerkin method for the first order MFG system

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Motivation for MFGs

- A game refers to a mathematical framework that models the interactions and behavior of a large number of **rational** agents, often referred to as players or agent.
- Each agent aims to optimize their own profit, taking into account the behavior of the other agents.
- Studying such systems $\Rightarrow$ differential games, through Nash equilibrium.
- Almost impossible to approximate the equilibria for games with large (big) number of agents.
- **MFG** $\Rightarrow$ Games with N number of players as $N \to \infty$.

Motivation: Many agent system

In today's interconnected world, systems involving numerous agents are prevalent.

Visual examples:

Crowd motion

Traffic flow

Flocking

Distributed AI systems

Other examples:

Markets

Financial market

Energy production

Networks

First order MFG system

The first order mean field game system is given by

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\begin{cases}
-\partial_t u + H(x, Du) = F(x, m(t)) & \text{in } (0,T)\times\mathbb{R}^d, \\
\partial_t m - \operatorname{div}(D_p H(x, Du) m) = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\
m(0,\cdot) = m_0^*, \quad u(T,x) = G(x, m(T)) & \text{in } \mathbb{R}^d.
\end{cases}
\tag{MFG$_1$}
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- The first equation is the Hamilton Jacobi Bellman equation for the agents′ value function u.
- The second equation is the continuity equation for the distribution of agents. $m(t)$ is the probability density of the state of players at time t
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- $m_0 \in \mathcal{P}(\mathbb{R}^d)$ can be seen as the initial distribution of the agents
- Solutions to $(\mathrm{MFG})_1$ are not regular in general, which makes the analysis more complicated.
- **Forward-Backward system**.

Lagrange-Galerkin method for the first order MFG system

- **Main idea**: Apply a semi-Lagrangian scheme to the HJB equation then couple it with a Lagrange-Galerkin scheme for the continuity equation.

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- **Assumptions**:
  - The Hamiltonian H is given by

$$H(x,p) = \sup_{a \in \mathbb{R}^d} \{-\langle a,p\rangle - L(x,a)\} \quad \text{for all } x,p \in \mathbb{R}^d,$$

where L is of class C^2, and for all $x,a \in \mathbb{R}^d$, we have

$$\begin{aligned} L(x,a) &\leq C(|a|^2+1), \\ |D_x L(x,a)| &\leq C(|a|^2+1), \\ C|b|^2 &\leq D^2_{aa} L(x,a)(b,b), \\ D^2_{xx} L(x,a)(y,y) &\leq C(|a|^2+1)|y|^2. \end{aligned}$$

These assumptions on L imply that H has quadratic growth and

$$|D_p H(x,p)| \leq C(1+|p|) \quad \text{for all } x,p \in \mathbb{R}^d.$$

A typical example is $H(x,p) = a(x)|p|^2 + \langle b(x),p\rangle$, with a and b of class C^2_b and a bounded from below by a strictly positive constant.

- F and G are bounded, continuous, and for every $\mu \in \mathcal{P}^1(\mathbb{R}^d)$,

$$\text{(Lip)} \quad |F(x,\mu) - F(y,\mu)| + |G(x,\mu) - G(y,\mu)| \leq C|x-y|,$$

$$\text{(SC)} \quad F(x+y,\mu) - 2F(x,\mu) + F(x-y,\mu) \leq C|y|^2,$$

$$\text{(SC)} \quad G(x+y,\mu) - 2G(x,\mu) + G(x-y,\mu) \leq C|y|^2.$$

  Notice that no differentiability is assumed for F and G.
- m_0^* has compact support and $m_0^* \in L^p(\mathbb{R}^d)$ for some $p \in (1,\infty]$.

Approximation to the HJB equation

Let $\mu \in C([0,T];\mathcal{P}_1(\mathbb{R}^d))$ and consider the HJB equation

$$-\partial_t u + H(x, Du) = F(x, \mu(t)) \quad \text{in } [0,T] \times \mathbb{R}^d,$$

$$u(T,x) = G(x, \mu(T)) \quad \text{in } \mathbb{R}^d.$$

If $u[\mu]$ denotes its solution, then for every $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$u[\mu](t,x) = \inf_{\alpha} \int_t^T \underbrace{L(\gamma(s),\alpha(s)) + F(\gamma(s),\mu(s))}_{\text{Running cost}}\, ds + \underbrace{G(\gamma(T),\mu(T))}_{\text{Final cost}}$$

$$\gamma \text{ satisfies } \dot{\gamma}(s) = -\alpha(s) \text{ in }]s,T[, \quad \gamma(t) = x.$$

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Proposition:

The value function is uniformly bounded, and the following hold:

$$\text{(Lip)} \quad |u[\mu](t,x) - u[\mu](t,y)| \le C|x-y|,$$

$$\text{(SC)} \quad u[\mu](x+y,\mu) - 2u[\mu](x,\mu) + u[\mu](x-y,\mu) \le C|y|^2.$$

Semi-Lagrangian scheme for HJB equation

- $u[\mu]$ satisfies the Dynamic Programming Principle:

$$u[\mu](t,x) = \inf_{\alpha \in L^2(\mathbb{R}^d)} \left\{ \int_t^{t+h} [L(\gamma(s), \alpha(s)) + F(\gamma(s), \mu(s)]\mathrm{d}s + u[\mu](t+h, \gamma(t+h)) \right\}$$

for all $h \in [0, T-t]$.

Semi-Lagrangian scheme for HJB equation

- Set $\Delta t > 0$ as the time step and let $t_k = k\Delta t$, $k = 0, \cdots N_T$.
- Semi-discrete DPP: let $u_k[\mu](x) \approx u[\mu](t_k, x)$ be such that

$$u_k[\mu](x) = \inf_{a \in \mathbb{R}^d} \Delta t[L(x,a) + F(x, \mu(t_k))] + u_{k+1}[\mu](x - a\Delta t)$$

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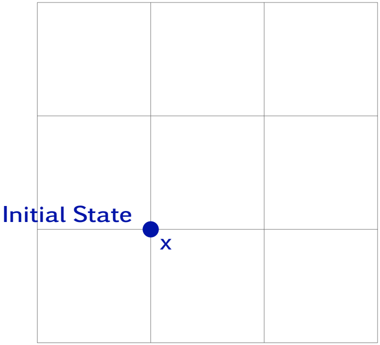
- Discretization in space: let $\Delta x > 0$ be the space step and let $\mathcal{G}_{\Delta x} = \{x_i = i\Delta x \,|\, i \in \mathbb{Z}^d\}$ be the grid space.



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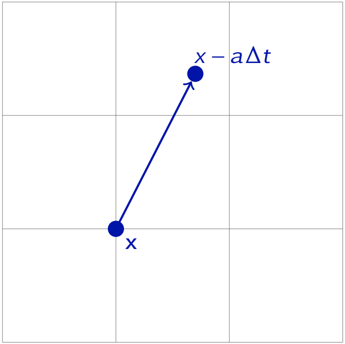

As in Carlini-S′14, given $(\Delta t, \Delta x)$ we consider the following **semi-Lagrangian** scheme for the HJB equation:

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\begin{aligned}
u_{k,i} &= \inf_{a \in \mathbb{R}^d} \left\{ \Delta t L(x_i, a) + I^1[u_{k+1,\cdot}](x_i - \Delta t a) \right\} + \Delta t F(x_i, \mu(t_k)), \\
u_{N,i} &= G(x_i, \mu(T)),
\end{aligned}
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where, given ϕ defined on $\mathcal{G}_{\Delta x} = \{x_i = \Delta x \,|\, i \in \mathbb{Z}^d\}$

$$
\mathrm{I}^1[\phi](x) = \sum_{i \in \mathbb{Z}^d} \beta_i^1(x) \phi(x_i), \quad \text{for all } x \in \mathbb{R}^d,
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where $\{\beta_i^1 \,|\, i \in \mathbb{Z}^d\}$ is the $\mathbb{Q}_1$-basis defined on the regular mesh $\mathcal{G}_{\Delta x}$.

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This scheme is shown to be consistent, stable, and preserves:

- (Lip) The Lipschitz property.
- (SC) The semiconcavity.

A regularized version

Given $\varepsilon > 0$ and a standard mollifier ρ_ε, we set $\Delta = (\Delta t, \Delta x, \varepsilon)$ and

$$u^\Delta[\mu](t,x) = (\rho_\varepsilon * I[u_k](x)) \qquad \text{for all } t \in [t_k, t_{k+1}), x \in \mathbb{R}^d.$$

- $u^\Delta[\mu]$ preserves the Lipschitz property.
- The following semi-concavity estimate holds:

$$\langle D^2_{xx} u^\Delta[\mu](t,x)y, y\rangle \leq C\left(1 + \left(\frac{\Delta x}{\varepsilon^2}\right)^2\right)|y|^2.$$

- **Theorem:** Under suitable assumptions on the parameters, if $\mu_n \to \mu$ and $\Delta_n \to 0$, then $u^{\Delta_n}[\mu_n] \to u[\mu]$ uniformly over compact sets, and $D_x u^{\Delta_n}[\mu_n] \to D_x u[\mu]$ a.e.

Approximation to the continuity equation

Let us consider the following continuity equation

$$\partial_t m - \mathrm{div}(D_p H(x, D_x u[\mu])m) = 0 \quad \text{in } (0,T)\times\mathbb{R}^d,$$
$$m(0) = m_0^*.$$

Using the properties of $u[\mu]$, one can show the existence of $m[\mu]$ solution to the continuity equation such that:

- $m[\mu](t,\cdot)$ has a compact support, independent of μ.
- The mass does not concentrate on final time

  $$\|m[\mu](t,\cdot)\|_{L^p} \le C\|m_0^*\|_{L^p}, \quad \text{for all } t\in(0,T).$$

  where C is independent of p.

To discretize the MFG system, we focus on

$$\partial_t m - \mathrm{div}(D_p H(x, D_x u^{\Delta}[\mu])m) = 0 \quad \text{in } (0,T) \times \mathbb{R}^d,$$
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Since u^Δ is smooth w.r.t state, this equation has a unique solution

$$m^\Delta[\mu](t,\cdot) = \Phi^\Delta[\mu](0,t,\cdot)\sharp m_0^*,$$

where $\Phi^\Delta[\mu](s,t,x)$ is the solution, at time t, of the ODE:

$$\dot\gamma(r) = -D_p H(\gamma(r), D_x u^\Delta[\mu](r,\gamma(r))) \quad \text{in } (s,T),$$
$$\gamma(s) = x.$$

Equivalently, for ϕ integrable with respect to $m^\Delta[\mu](s)$,

$$\int_{\mathbb{R}^d} \phi(x)\, dm^\Delta[\mu](t)(x) = \int_{\mathbb{R}^d} \phi(\Phi^\Delta[\mu](s,t,x))\, dm^\Delta[\mu](s)(x) \tag{CE}$$

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- We approximate $\Phi^\Delta[\mu](t_k, t_{k+1}, x)$ by explicite one-step Euler scheme

$$\Phi_k^\Delta[\mu](x) = x - \Delta t D_p H(x, D_x u^\Delta[\mu](t_k, x)).$$

- Let $\{\beta_i\}_{i\in\mathbb{Z}^d}$ be a FE basis and approximate $m^\Delta[\mu](t_k)$ by

$$\mathsf{M}^\Delta[\mu](t_k, x) = \sum_{i\in\mathbb{Z}^d} m_{k,i}\beta_i(x).$$

- Using this approximation and taking $\phi = \beta_j$ in (CE), we get

$$\sum_{i\in\mathbb{Z}^d} m_{k+1,i}\int_{\mathbb{R}^d}\beta_i(x)\beta_j(\Phi_k^\Delta[\mu](x))\,\mathrm{d}x = \sum_{i\in\mathbb{Z}^d} m_{k,i}\int_{\mathbb{R}^d}\beta_j(\Phi_k^\Delta[\mu](x))\beta_i(x)\,\mathrm{d}x.$$

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- Let us choose $\beta_i = \beta_i^0 = \mathbb{I}_{E_i}$, where

$$E_i = [x_i - \Delta x/2, x_i + \Delta x/2]^d.$$

This choice yields the following Lagrange-Galerkin scheme:

$$m_{k+1,i} = \frac{1}{(\Delta x)^d}\sum_j m_{k,j}\int_{E_j}\beta_i^0(\Phi_k^\Delta[\mu](x))\,\mathrm{d}x \tag{LG}$$

$$m_{0,i} = \frac{1}{(\Delta x)^d}\int_{E_i} m_0^*(x)\,\mathrm{d}x.$$

The scheme preserves the properties

- Given $(m_{k,i})$ solution to (LG), for $t \in [t_k, t_{k+1})$, let us define

$$\mathbf{M}^{\Delta}[\mu](t,x) = \left(\frac{t_{k+1}-t}{\Delta t}\right) \sum_{i \in \mathbb{Z}^d} m_{k,i} \beta_i(x) + \left(\frac{t-t_k}{\Delta t}\right) \sum_{i \in \mathbb{Z}^d} m_{k+1,i} \beta_i(x).$$

- $\mathbf{M}^{\Delta}[\mu] \in C([0,T]; \mathcal{P}^1(\mathbb{R}^d))$.
- There exists $C^* > 0$ such that $\operatorname{supp}(\mathbf{M}^{\Delta}[\mu](t,\cdot)) \subseteq B(0, C^*)$.
- The map $[0,T] \ni t \mapsto \mathbf{M}^{\Delta}[\mu](t,\cdot) \in \mathcal{P}^1(\mathbb{R}^d)$ is Lipschitz continuous.
- If $\Delta x = O(\Delta t)$ and $\Delta t = O(\varepsilon^2)$, then

$$\|\mathbf{M}^{\Delta}[\mu](t,\cdot)\|_{L^p} \leq C \|m_0^*\|_{L^p}.$$

The proof of the L^p-stability mainly relies on the following facts:

- $\Delta t / \varepsilon$ small enough $\Rightarrow \Phi_k^{\Delta}[\mu]$ is one-to-one.
- The estimate on $D_{xx}^2 u^{\Delta}[\mu](t_k,\cdot)$ implies that

$$\det(D_x \Phi_k^{\Delta}[\mu](x))^{-1} \leq 1 + C\Delta t.$$

Approximation of the MFG problem

Let $u^{\Delta}[\mu]$ be the solution to the SL scheme and $\mathbf{M}^{\Delta}$ the solution to the LG scheme, then:

- $(\mathrm{MFG})_1$ is discretized as follows:

$$\text{Find } \mu \text{ such that } \mu = \mathbf{M}^{\Delta}[\mu] \qquad (\mathrm{MFG})^{\Delta}.$$

  Using the Brouwer's fixed point theorem, we show that $(\mathrm{MFG})^{\Delta}$ admits at least one solution.

- Convergence holds in general state dimensions.

Theorem (Carlini-Silva-Z'23)

Let $\Delta_n = (\Delta t_n, \Delta x_n, \varepsilon_n) \in]0,\infty[^3$, let m^n be a solution to $(MFG)^{\Delta_n}$, and $u^n = u^{\Delta}[m_n]$. Assume that, as $\Delta_n \to 0$, $\Delta x_n = o(\Delta t_n)$ and $\Delta t_n = O(\varepsilon_n^2)$. Then, up to some subsequence, (u^n, m^n) converges to a solution (u^*, m^*) of $(MFG)_1$.

Numerical results

- In order to implement the scheme, we follow Morton-Priestley-Süli'88 by considering the following approximation called area weighting

$$\Phi_k^{\Delta}[\mu](x) \approx x - \Delta t D_p H(x_i, D_x u^{\Delta}[\mu](t_k, x_i)) \quad \text{if } x \in E_i,$$

to obtain

$$\int_{E_j} \beta_i^0(\Phi_k^{\Delta}[\mu](x))\mathrm{d}x = \beta_i^1(\Phi_k^{\Delta}[\mu](x_j)).$$

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- In the numerical test below, we set $d = 2$, and we consider the MFG problem defined on $[0,1] \times [0,2]^2$, and set $\Delta t = (\Delta x)^{\frac{2}{3}}$

$$m_0^*(x) = \frac{\nu(x)}{\int_{[0,2]^2} \nu(x)dx} \mathbb{I}_{[0,2]^2} \quad \text{with } \nu(x) = e^{-|x-x_0|^2/0.01} \quad \text{and } x_0 = (0.75, 0.75).$$

We also consider

$$H(x,p) = \frac{|p|^2}{2}, \quad G = 0$$

and

$$F(x,m) = \underbrace{\gamma \min(R, |x - x_f|^2)}_{\text{penalize the deviation from } x_f} + \underbrace{(\rho_\sigma * m)(x)}_{\text{encourage avoiding the crowd}}$$

with $x_f = (1.75, 1.75)$.

In the figures below, we display the distributions for $\gamma = 1$ and $\gamma = 3$.

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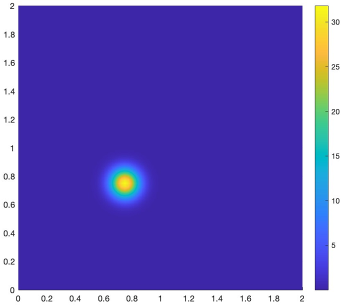
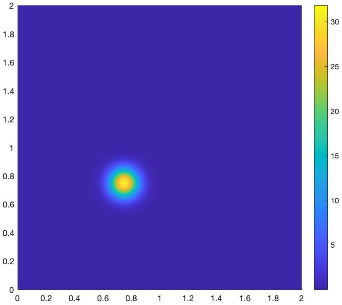


$m(0,x),\ \gamma = 1$



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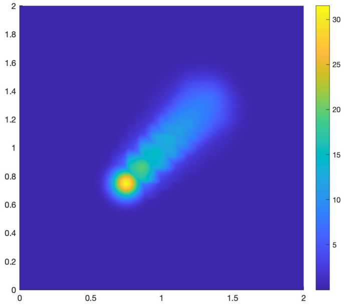
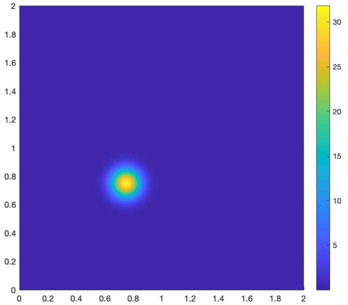


The evolution of m, $\gamma = 1$



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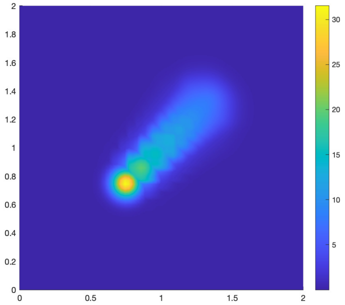
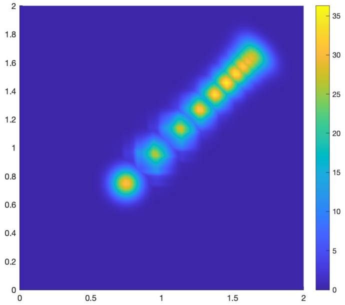
In the figures below, we display the distributions for $\gamma = 1$ and $\gamma = 3$.



The evolution of m, $\gamma = 1$



The evolution of m, $\gamma = 3$

References

- E. Carlini, F. J. Silva, and A. Zorkot, A Lagrange–Galerkin scheme for first order mean field game systems, SIAM Journal on Numerical Analysis, 62 (2024)
- F. Camilli and F. J. Silva. A semi-discrete in time approximation for a first order-finite mean field game problem. Network and Heterogeneous Media, 2012.
- E. Carlini and F. J. Silva. A fully discrete semi-Lagrangian scheme for a first order mean field game problem. SIAM J. Numer. Anal.
- Y. Achdou and M. Laurière. Mean field games and applications: numerical aspects. Lecture Notes in Math., 2020.
- J. Gianatti and F. J. Silva. Approximation of deterministic mean field games with control-affine dynamics. Foundations of Computational Mathematics, 2022.
- J. Gianatti, F. J. Silva, and A. Zorkot. Approximation of deterministic mean field games under polynomial growth conditions on the data. Journal of dynamics and games, 2023.