Estimation of the effective sound speed in an acoustic medium

Josselin Garnier, Laure Giovangigli, Quentin Goepfert and Pierre Millien

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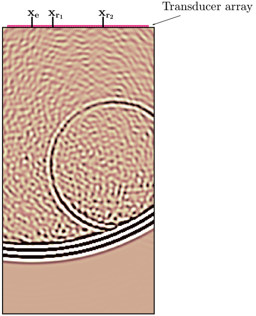
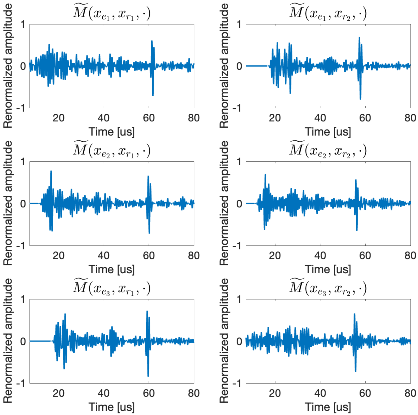
Introduction - Principle of ultrasound imaging





Measurements

We have access for the frequency $\omega \in \mathcal{B}$ (the bandwidth), $\boldsymbol{x_e}, \boldsymbol{x_r} \in \mathcal{A}$ (the transducer array) to the **reflection matrix**:

$$M(\boldsymbol{x_e}, \boldsymbol{x_r}, \omega).$$

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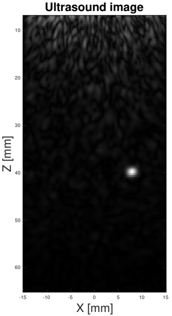

Figure: Simulated image

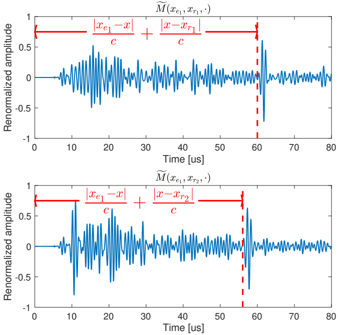

Figure: Delay for the computation of the image

Kirchhoff imaging function

$$\mathcal{I}(\boldsymbol{x_s}, c_s) = \int_{\mathcal{B}\times\mathcal{A}\times\mathcal{A}} \overline{M}(\boldsymbol{x_e}, \boldsymbol{x_r}, \omega) G^{\frac{\omega}{c_s}}(\boldsymbol{x_e}, \boldsymbol{x_s}) G^{\frac{\omega}{c_s}}(\boldsymbol{x_s}, \boldsymbol{x_r}) d\boldsymbol{x_e} d\boldsymbol{x_r} d\omega$$

$G^k(\boldsymbol{x}, \cdot)$ is the outgoing Green function of the Helmholtz equation at wavenumber k for a source in $\boldsymbol{x}$.

Introduction - Assumptions and restrictions in the reconstruction domain

Two assumptions are done in the reconstruction model:

- The speed of sound is known and constant throughout the medium.
- The echoes come only from single scattering (Born approximation).

These two assumptions are strong and restrictive in practical situations.

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Speed of sound estimation

- The speed of sound drives the quality of the image and the fidelity of the tomographic reconstruction.
- It is a quantitative biomarker for diagnosis (breast cancer, hepatic steatosis...).

State of the art - speed of sound estimation

Existing methods - access to the transmitted field

Inversing the eikonal equation - Extensive mathematical analysis...

Existing methods - reflection only

- **Spatial coherence methods**: Compare the image produced with several subsets of the sensors array and compute the spatial coherence [O'Donnell 01, Jin Ho Chang 12, ...].
  Requires more hardware processing but local estimations.
- **Compounding methods**: Cross correlate the images computed for several incident angles (see for example the CUTE method) [Frenz 15,Goksel 21, ...].
  Low robustness but local (and direct) estimations.
- **Focusing methods**: Adjust the speed of sound to maximize the quality of the image (described by the lateral resolution or the echo amplitude) [Umemura 88, Ogawa 19, Aubry 23, ...].
  Usually non local estimations but easy to implement.

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Experimental theories but few mathematical modeling...

Outline

1. Point spread function in a homogeneous medium

2. Estimation of the effective speed of sound in the homogenization regime

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2. Estimation of the effective speed of sound in the homogenization regime

Point spread function (PSF)

The PSF is the response of an imaging system to a point like reflector.
The PSF is known when the backpropagation speed matches the speed of sound of the medium [Papanicolaou 07].

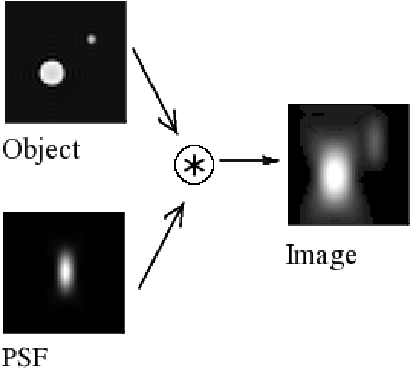

Figure: Source: Wikipedia

What are the effects of a mismatch ?

Simulated images for several speed of sound

What are the effects of a mismatch ?

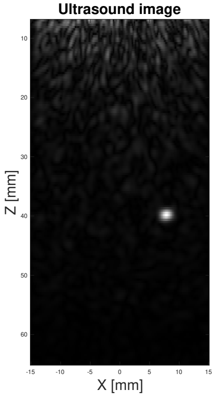

Figure: Simulated ultrasound image with guessed speed $c = 1500$ m.s^{-1}

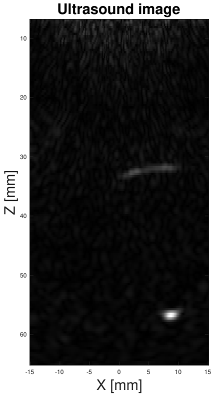

Figure: Simulated ultrasound image with guessed speed $c = 1200$ m.s^{-1}

Measurements

For a given frequency $\omega \in \mathcal{B}$ (the bandwidth), we consider spherical incident waves u^i emitted in $\boldsymbol{x_e} \in \mathcal{A}$ (the transducer array),

$$u^i(\boldsymbol{x_e}, \boldsymbol{x}, \omega) := G^{\frac{\omega}{c_0}}(\boldsymbol{x_e}, \boldsymbol{x}).$$

The scattered field u^s is the unique solution in $H^1_{loc}(\mathbb{R}^d)$ of:

Equation verified by u

$$\begin{cases} -\Delta(u^s + u^i)(\boldsymbol{x_e}, \cdot, \omega) - \dfrac{\omega^2}{{c_0}^2}(1 + n(\cdot))(u^s + u^i)(\boldsymbol{x_e}, \cdot, \omega) = 0 \quad \text{in } \mathbb{R}^d \\ \lim\limits_{|\boldsymbol{x}| \to \infty} |\boldsymbol{x}|^{\frac{d-1}{2}} \left(\dfrac{\partial}{\partial |\boldsymbol{x}|} u^s(\boldsymbol{x_e}, \boldsymbol{x}, \omega) - i \dfrac{\omega}{c_0} u^s(\boldsymbol{x_e}, \boldsymbol{x}, \omega) \right) = 0 \end{cases}$$

where $n \in L^\infty(\mathbb{R}^d)$, and $\mathrm{supp}(n) \subset D$ and $(1 + n) > 0$. $D \subset \mathbb{R}^d$ is the domain to image. c_0 is the unknown speed to recover.

We have access for all $\boldsymbol{x_e}, \boldsymbol{x_r} \in \mathcal{A}$, $\omega \in \mathcal{B}$ to the measurement map

$$M(\boldsymbol{x_e}, \boldsymbol{x_r}, \omega) = u^s(\boldsymbol{x_e}, \boldsymbol{x_r}, \omega)$$

Numerical simulations - PSF for several backpropagation speeds

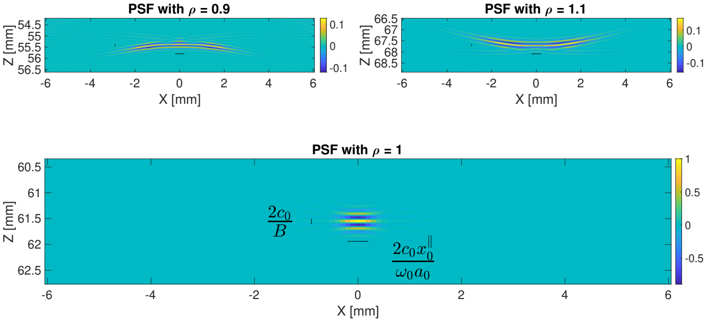

Figure: Point spread function for several backpropagation speeds.

Notations:

- $\rho := \frac{c_0}{c_s}$.
- a_0: size of the sensors array.
- ω_0: central frequency.
- $x_0^{\parallel}$: depth.
- B: bandwidth.

The **shape**, the **position of the target** on the image and the **amplitude** is altered by a mismatch.

Speed of sound estimation

For a given c_s, the brightest pixel at $x^\star(c_s)$ on the image has the behavior

$$\mathcal{I}(x^\star(c_s), c_s) \sim \mathcal{G}\left(0, \frac{a_0^2\omega_0}{x_0^{\shortparallel} c_0}(\rho^2 - 1)\right) := \frac{1}{2}\int_{-1}^{1} \exp\left(i\frac{|\boldsymbol{x_e}|^2}{2}\frac{a_0^2\omega_0}{x_0^{\shortparallel} c_0}(\rho^2 - 1)\right) d\boldsymbol{x_e}.$$

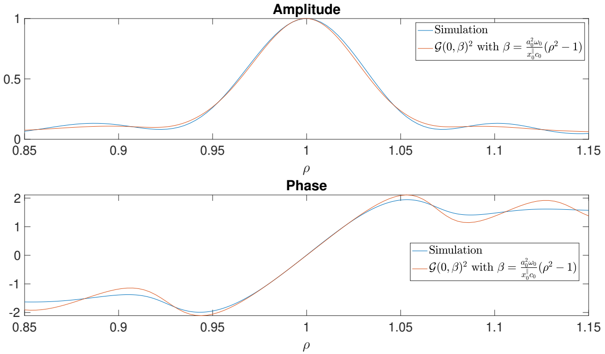

Figure: PSF in the bright pixel - Theoretical and simulation results

The speed of sound of the medium is estimated by $\hat{c}_0 := \underset{c_s}{\operatorname{argmax}} |\mathcal{I}(x^\star(c_s), c_s)|$.

Multiscale random medium

In practical situations, there is no reflector.
The medium is composed of **numerous unresolved scatterers**. It can be modeled as **random**.

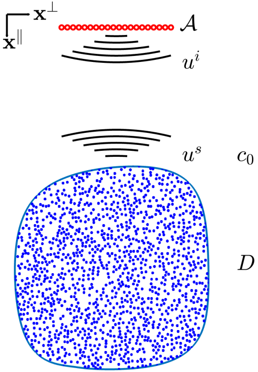

Figure: Schema of the homogenization model

Two difficulties:

- Born approximation does not hold
  How to characterize the scattered field in this regime?
- Absence of target
  How to recover the effective speed of sound ?

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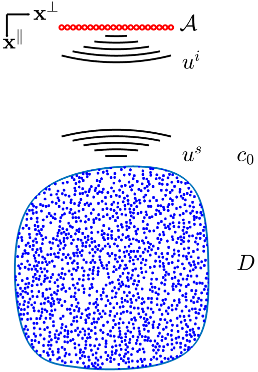

Figure: Schema of the homogenization model

Two difficulties & two answers :

- Born approximation does not hold
  How to characterize the scattered field in this regime?
  Stochastic homogenization
- Absence of target
  How to recover the effective speed of sound ?
  Local spatial average of the imaging function

Scattered field in a multiscale random medium

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The scattered field u^s is the solution of:

Equation verified by u

$$\begin{cases} -\Delta(u^s + u^i)(\boldsymbol{x_e}, \cdot, \omega) - \dfrac{\omega^2}{{c_0}^2}(1 + n(\cdot))(u^s + u^i)(\boldsymbol{x_e}, \cdot, \omega) = 0 \quad \text{in } \mathbb{R}^d \\ \lim\limits_{|\boldsymbol{x}| \to \infty} |\boldsymbol{x}|^{\frac{d-1}{2}} \left(\dfrac{\partial}{\partial |\boldsymbol{x}|} u^s(\boldsymbol{x_e}, \boldsymbol{x}, \omega) - i \dfrac{\omega}{c_0} u^s(\boldsymbol{x_e}, \boldsymbol{x}, \omega) \right) = 0 \end{cases}$$

where for $\boldsymbol{x} \in D$, $n(\boldsymbol{x}) = \tilde{n}(\frac{\boldsymbol{x}}{\eta})$ where $\tilde{n} \in L^2(\Omega, L^\infty(\mathbb{R}^d))$ is a random stationary and ergodic process.
$\eta \ll 1$ corresponds to the typical ratio of the size of the scatterers over the wavelength.

Stationarity: the distribution of $\tilde{n}$ does not change when shifted.
Ergodicity:

$$\forall \boldsymbol{x} \in \mathbb{R}^d, \ \frac{1}{|B(0,R)|} \int_{B(0,R)} \tilde{n}(\boldsymbol{x} + \boldsymbol{y}) dy \xrightarrow[R \to \infty]{} \mathbb{E}[\tilde{n}(0)].$$

Asymptotic approximation of the scattered field

In the stochastic homogenization regime, under quantitative mixing assumptions[1]:

Asymptotic approximation of the scattered field

For all $\boldsymbol{x_e}, \boldsymbol{x_r} \in \mathcal{A}$, $\omega \in \mathcal{B}$,

$$u^s(\boldsymbol{x_e}, \boldsymbol{x_r}, \omega) = (\frac{\omega}{c_0})^2 \int_D \left(\tilde{n}(\frac{\boldsymbol{x}}{\eta}) - \mathbb{E}[\tilde{n}(0)] \right) G_*^{\frac{\omega}{c_\star}}(\boldsymbol{x_e}, \boldsymbol{x}) G_*^{\frac{\omega}{c_\star}}(\boldsymbol{x}, \boldsymbol{x_r}) d\boldsymbol{x} + O(\eta^{\frac{d+1}{2}})$$

with

$$c_\star{}^2 := \frac{c_0{}^2}{1 + \mathbb{E}[\tilde{n}(0)]}.$$

and $G_\star^{\frac{\omega}{c^\star}}$ (the Green function of the homogenized medium) verifies:

$$\begin{cases} -\Delta G_\star^{\frac{\omega}{c^\star}}(\boldsymbol{x_e}, \cdot) - \dfrac{\omega^2}{c_0{}^2}(1 + \mathbb{E}[n(\cdot)]) G_\star^{\frac{\omega}{c^\star}}(\boldsymbol{x_e}, \cdot) = \delta(\boldsymbol{x_e}) \quad \text{in } \mathbb{R}^d \setminus \{\boldsymbol{x_e}\} \\ \lim\limits_{|\boldsymbol{x}| \to \infty} |\boldsymbol{x}|^{\frac{d-1}{2}} \left(\dfrac{\partial}{\partial |\boldsymbol{x}|} G_\star^{\frac{\omega}{c^\star}}(\boldsymbol{x_e}, \cdot) - i \dfrac{\omega}{c_0} G_\star^{\frac{\omega}{c^\star}}(\boldsymbol{x_e}, \cdot) \right) = 0 \end{cases}$$

[1] Scattered wavefield in the stochastic homogenization regime Josselin Garnier, Laure Giovangigli, Q. G., Pierre Millien, *submitted*

Imaging function

Imaging function in the stochastic homogenization regime

Let $\boldsymbol{x_s} = (\boldsymbol{x_s^\perp}, x_s^{\shortparallel}) \in D$. Then the imaging function $\mathcal{I}$ has the following form:

$$\mathcal{I}\left(\boldsymbol{x_s}, c_s = \frac{c_\star}{\rho_\star}\right) \approx \int_{\mathbb{R}^d} \left(\tilde{n}\left(\frac{(\rho_\star^2 \boldsymbol{x_s^\perp}, \rho_\star x_s^{\shortparallel}) + \boldsymbol{x}}{\eta}\right) - \mathbb{E}[\tilde{n}(0)]\right) \mathcal{P}(\boldsymbol{x}, \boldsymbol{x_s}, \rho_\star) d\boldsymbol{x}.$$

where $\mathcal{P}$ is the previous point spread function.

At first order for $\eta \ll \boldsymbol{\Delta x_s} \sim \lambda := \frac{c_\star}{\omega_0}$,

$$\mathcal{P}(\cdot, \boldsymbol{x_s} + \boldsymbol{\Delta x_s}, \rho_\star) \approx \mathcal{P}(\cdot, \boldsymbol{x_s}, \rho_\star).$$

but

$$\tilde{n}\left(\frac{(\rho_\star^2(\boldsymbol{x_s^\perp} + \boldsymbol{\Delta x_s^\perp}), \rho_\star(x_s^{\shortparallel} + \Delta x_s^{\|}))}{\eta}\right) \not\approx \tilde{n}\left(\frac{(\rho_\star^2 \boldsymbol{x_s^\perp}, \rho_\star x_s^{\shortparallel}}{\eta}\right)$$

and therefore for a given $r > 0$ (of the order of the wavelength),

$$\mathcal{I}^{av}(\boldsymbol{x_s}, c_s) = \int_{B(0,r)} |\mathcal{I}(\boldsymbol{x_s} + \boldsymbol{y_s}, c_s)|^2 d y_s \to \mathbb{E}\left[|\mathcal{I}(\boldsymbol{x_s}, c_s)|^2\right] \propto \int_{\mathbb{R}^d} \left|\mathcal{P}(\boldsymbol{x}, \boldsymbol{x_s}, \rho_\star)\right|^2 d\boldsymbol{x}$$

Numerical illustrations - Medium in the stochastic homogenization regime

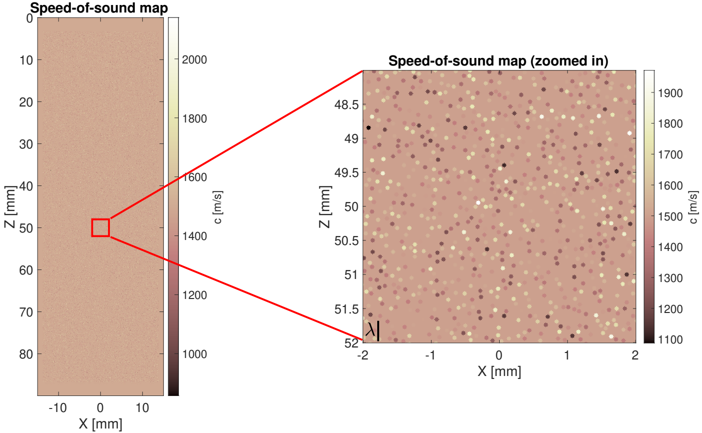

Figure: Sound speed map a random multiscale medium

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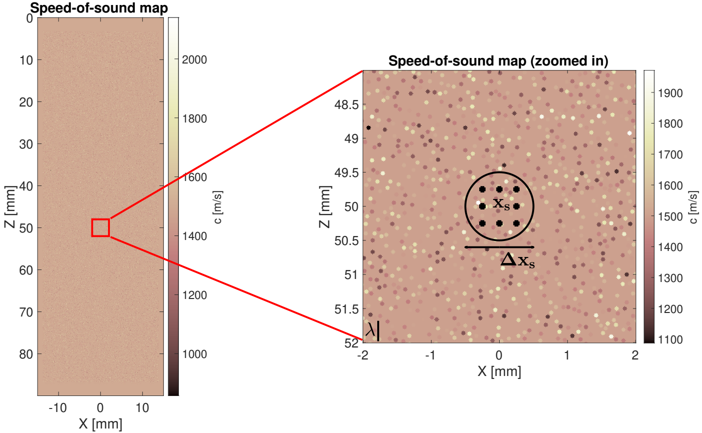

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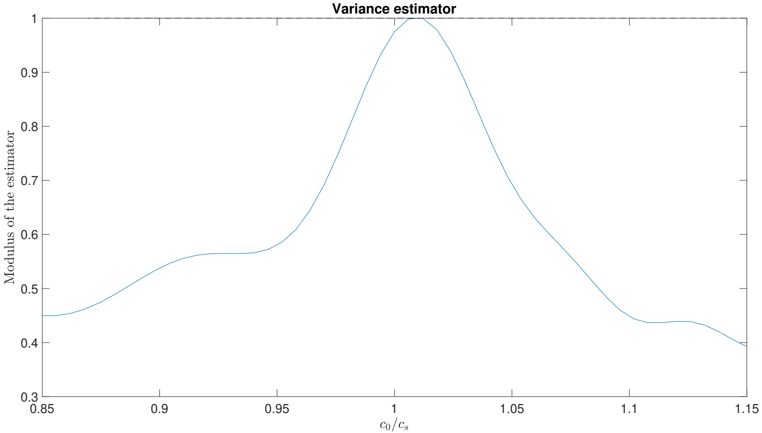

The speed of sound of the medium is estimated by

$$\hat{c}_\star := \underset{c_s}{\operatorname{argmax}} \, \mathcal{I}^{av}(\boldsymbol{x_s}, c_s).$$

Conclusion & Perspectives

Conclusion & Perspectives

So far, we have:

- expressed the PSF when the backpropagation speed does not match the speed of the medium in a homogeneous medium
- established an estimator of the speed of sound in a homogeneous medium with isolated target.
- established an estimator of the effective speed of sound in a random multiscale medium.

Perspectives:

- extend the method to more realistic situations (slowly varying effective speed of sound)
- extend the method to inhomogeneous media with contrast both in the bulk modulus and density.

Conclusion & Perspectives

Thank you for your attention!