Numerical integration for the nonlinear Klein–Gordon equations in low regularity and conservation properties

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We consider the nonlinear Klein–Gordon equations with periodic boundary conditions

$$\partial_{tt} q + (m - \partial_{xx}) q + f(q) = 0 \tag{KG}$$

where $(x,t) \in \mathbb{T} \times \mathbb{R}$, $q = q(x,t) \in \mathbb{R}$, the mass $m > 0$ and f is a smooth real non-linearity with $f(0) = f'(0) = 0$. We set $p = \partial_t q$, and we assume that the initial data $(q_{|t=0}, p_{|t=0}) = (q(0), p(0))$ is small in $H^s \times H^{s-1}$. Close to the origin, the (KG) equations are nearly integrable Hamiltonian systems

$$\partial_t \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \partial_{xx} - m & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla H(q,p)$$

having infinitely many almost conserved quantities $\mathcal{E}_j$ called "harmonic energies" or "super-actions" :

$$\mathcal{E}_j(q,p) = |\hat{q}(j)|^2 + \omega_j^{-2} |\hat{p}(j)|^2$$

where $\omega_j = (j^2 + m)^{1/2}$ are the eigenvalues of operator $(-\partial_{xx} + m)^{1/2}$. In a series of works in 2008, D. Cohen, E. Hairer and C. Lubich proved that, at high regularity (s very large), classical symplectic numerical integrators preserve this qualitative property with CFL of size 1. On the other hand, their numerical simulations strongly suggested that the smoothness assumption is irrelevant and can be relaxed [3]. In this talk, I will present a work in which we extend their result for non-smooth solutions (s=1). The main result is given as follow :

We consider a numerical solution (q^n, p^n) obtained by applying a pseudo-spectral semi discretization with K equidistant collocation points, and a symplectic numerical integrator to (KG). Then,

Théorème 1. *Let $\delta \in (0, 2\pi)$. There exist $\beta_r > 0$ and $\varepsilon_m > 0$ such that for almost all mass $m > 0$ and all $r \geq 1$ arbitrarily large, provided that $s = 1$, $||q^0, p^0||_{H^1 \times L^2} := \varepsilon < \varepsilon_m$ and $rh\omega_{(K-1)/2} \leq 2\pi - \delta$, the numerical flow of (KG) satisfies*

$$nh < \varepsilon^{-r} \implies \sum_{|j| < K/2} \langle j \rangle^{-2\beta_r} \frac{|\mathcal{E}_j(q^n, p^n) - \mathcal{E}_j(q^0, p^0)|}{\varepsilon^2} \lesssim_{r,m,\delta} \varepsilon.$$

The proof is based on Splitting methods, backward error analysis and Birkhoff normal form in low regularity [2]. It will be found in the paper (in progress), in collaboration with Joackim Bernier [1].

Références

[1] Abou Khalil, C & Bernier, J. Numerical integration for the nonlinear Klein–Gordon equations in low regularity and conservation properties, *in preparation*

[2] Bernier, J. & Grébert, B. Birkhoff normal forms for Hamiltonian PDEs in their energy space, Journal de l'Ecole polytechnique, Tome 9 (2022), pp. 681-745.

[3] Cohen, D., Hairer, E. & Lubich, C. Conservation of energy, momentum and actions in numerical discretizations of non-linear wave equations. *Numer. Math.*. **110**, 113-143 (2008), https ://doi.org/10.1007/s00211-008-0163-9

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