Local density interpolation applied to boundary integral methods

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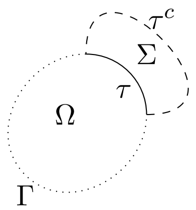
In this exposition we present a technique to deal with singular integrals ubiquitous in boundary integral equation methods. In most applications they are of the following form :

$$\alpha \int_\Gamma \gamma_{1,\boldsymbol{y}} G(\boldsymbol{x},\boldsymbol{y})\sigma(\boldsymbol{y})\mathrm{d}s_{\boldsymbol{y}} - \beta \int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\sigma(\boldsymbol{y})\mathrm{d}s_{\boldsymbol{y}},\ \boldsymbol{x} \in \Gamma, \tag{1}$$

where Γ is the boundary of a domain in $\mathbb{R}^d$, G is the free-space Green's function, γ_1 the Neumann trace operator of the elliptic differential operator in question, and α, β are constants. The idea is to employ Green's identity to transport singular integrals to a new boundary without singularity, allowing for the use of standard quadrature rules. Unlike previous works on density interpolation methods (e.g. [2]), we focus here on the construction of *local interpolants* which need only satisfy the underlying partial differential operator on a small neighborhood of the singularity. This alleviates some issues encountered in the global density interpolation method, such as the need of a global interpolation basis, making it simpler to add various levels of *local refinement*.
Let $\tau \subset \Gamma$ be a patch containing the singular point $\boldsymbol{x}$, and let $\Sigma \subset \mathbb{R}^d$ be a domain such that $\tau \subset \partial\Sigma$ (see Figure 1). The *local density interpolation* problem consists of finding a function $\tilde{\sigma}$ such that



$$\begin{cases} \mathcal{L}\tilde{\sigma} = 0, & \text{in } \Sigma; \\ \tilde{\sigma} = \alpha\sigma, & \text{on } \tau; \\ \gamma_1\tilde{\sigma} = \beta\sigma, & \text{on } \tau. \end{cases} \tag{2}$$

FIGURE 1 – Schematic of interpolation domain

where $\mathcal{L}$ is the underlying elliptic operator (e.g. $\mathcal{L} = \Delta$). This method is in principle applicable to a wide range of PDEs of physical interest. In this exposition, a semi-analytic construction procedure is developed for the 2-dimensional Laplace and Stokes equations thanks to the expression of their solutions in complex variables (c.f. for example [1]).

[1] M. Kropinski. *An efficient numerical method for studying interfacial motion in two-dimensional creeping flows.* Journal of Computational Physics, **171(2)**, 479–508, 2001.

[2] C. Pérez-Arancibia, L. M. Faria, C. Turc. *Harmonic density interpolation methods for high-order evaluation of laplace layer potentials in 2d and 3d.* Journal of Computational Physics, **376**, 411–434, 2019.

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