Growth conditions to ensure input-to-state stability of time-delay systems with point-wise dissipation

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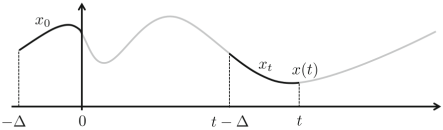
Definition 1

A **time-delay system** (TDS) is a system modeled by:

$$\dot{x}(t) = f(x_t, u(t)). \tag{1}$$

$u(t) \in \mathbb{R}^m$ is the input
$x_t : [-\Delta, 0] \to \mathbb{R}^n$ is the solution's history defined by $x_t(s) = x(t+s)$ for all $s \in [-\Delta, 0]$,



Δ is the maximum delay and we assume that f is Lipschitz on bounded set in what follows.

Comparision functions

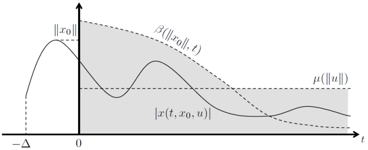
1. Class $\mathcal{K}_\infty$ function: zero at zero, continuous, increasing and unbounded.
2. Class $\mathcal{KL}$ function : $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$
   - $\beta(\cdot, t)$ is zero at zero, continuous and increasing
   - $\beta(s, \cdot)$ is continuous, decreasing and vanishing at infinity.

Figure: Class $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions.

Definition 2

TDS (1) is said to be **input-to-state stable (ISS)** if there exist $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_\infty$ such that, for all $x_0 \in C([-\Delta, 0], \mathbb{R}^n)$ and all $u \in L^\infty_{loc}(\mathbb{R}^+, \mathbb{R}^m)$,

$$|x(t, x_0, u)| \leq \beta(||x_0||, t) + \mu(||u_{[0,t]}||), \quad \forall t \geq 0. \tag{2}$$



If $\beta(s,t) = kse^{-\lambda t}$, the TDS (1) is said to be **exponentially input-to-state stable (exp-ISS)**.

Definition 3

A functional $V : C([-\Delta, 0], \mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ is said to be a **Lyapunov-Krasovskii functional candidate (LKF)** if it is **Lipschitz on bounded sets** and there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that, for all $t \geq 0$

$$\underline{\alpha}(|x(t)|) \leq V(x_t) \leq \overline{\alpha}(\|x_t\|). \quad (3)$$

$|x(t)|$ is a $\mathbb{R}^n$ norm which differs from $\|x_t\| := \sup_{\tau \in [-\Delta, 0]} |x_t(\tau)|$.

Dissipations of LKF

Definition 4

For TDS (1), an LKF V is said to be

1. an ISS LKF with LKF-wise dissipation if there exist $\alpha, \gamma \in \mathcal{K}_\infty$ such that

$$D^+V \leq -\alpha(V(x_t)) + \gamma(|u(t)|), \tag{4}$$

2. an ISS LKF with point-wise dissipation if there exist $\alpha, \gamma \in \mathcal{K}_\infty$ such that

$$D^+V \leq -\alpha(|x(t)|) + \gamma(|u(t)|). \tag{5}$$

D^+V is the Driver's derivative of the functional V along the solution of (1).

Remarks

- From (3), the following implication holds:

$$(4) \Rightarrow (5)$$

- The dissipation rate α involves $|x(t)|$ in (5) (so easier) and not the LKF V itself as in (4).

[Karafyllis et al., 2008]

Theorem 5

TDS (1) is ISS if and only if it admits an ISS LKF with LKF-wise dissipation.

A way to get LKF-wise dissipation

Consider the LKF

$$V(x_t) = V_1(x(t)) + \int_{-\Delta}^{0} V_2(x_t(s))ds, \tag{6}$$

Usually, according to the expression of f, the derivative of V can easily look like

$$D^+V \leq -\alpha(|x(t)|) + \gamma(|u(t)|).$$

By adding ke^{cs} in the kernel of integral part of (6), we have

$$W(x_t) = V_1(x(t)) + \int_{-\Delta}^{0} ke^{cs} V_2(x_t(s))ds,$$

and with $k > 1$

$$D^+W \leq -\alpha(|x(t)|) - (1-k)V_2(x(t)) - kc\int_{-\Delta}^{0} ke^{cs} V_2(x_t(s))ds + \gamma(|u(t)|).$$

When the "exponential trick" works

[Loko et al., 2024](CDC)

Proposition 1

If there exists a constant $p > 0$ *such that*

$$\alpha(|x(t)|) \geq pV_2(x(t)),$$

then (9) *is an ISS LKF with LKF-wise dissipation and the TDS* (1) *is ISS.*

Proposition 1 extends the result given in [Orłowski et al., 2022, Lemma 1] to non necessarly quadratic LKFs.

When the "exponential trick" does not work

Consider the following **1D** TDS

$$\dot{x}(t) = -x(t) - \frac{x(t)}{1+x(t)^2} + \frac{x(t-1)^4}{1+|x(t)|^3} + \frac{u(t)}{1+x(t)^2}, \tag{7}$$

and the LKFs:

$$V(x_t) := \frac{1}{4}x(t)^4 + \int_{-1}^{0} x_t(s)^4 ds,$$

$$W(x_t) := V_1(x(t)) + \int_{-\Delta}^{0} ke^{cs} V_2(x_t(s))ds.$$

[Loko et al., 2024](CDC)

Proposition 2

1. *System* (7) ***is ISS***.
2. V *is an ISS LKF with* ***point-wise dissipation*** *for* (7)
3. *Given any* $k, c > 0$, W *is not an LKF with LKF-wise dissipation.*

Getting ISS with point-wise dissipation?

[Chaillet et al., 2017]

Conjecture 1

Assume that there exist an LKF V with point-wise dissipation meaning (5) *holds. Then the system* (1) *is ISS.*

In response,

1. [Chaillet et al., 2017] provides a sufficient condition on the growth of the vector field of TDS.
2. [Chaillet et al., 2023] provides some growth conditions on the LKF to ensure exp-ISS.

[Loko et al., 2024](CDC)

Theorem 6

Assume that there exists a LKF V which dissipates point-wisely as follow,

$$D^+V \leq -\alpha(Q(x(t))) + \gamma(|u(t)|). \tag{8}$$

Assume that

$$\dot{Q}(x(t)) \leq \sigma(\|Q\|) + \gamma(|u(t)|). \tag{9}$$

Then, if

$$\liminf_{r \to +\infty} \frac{\alpha(r)}{\sigma(re^{2\Delta})} > 0, \tag{10}$$

the system (1) *is ISS.*

Comments

- (8) shows the point-wise dissipation of V
- (9) presents the growth of the function Q along the solution of the system
- (10) requires the growth on Q to be smaller than the dissipation rate α (it is the main requirement)
- Theorem 6 generalizes [Chaillet et al., 2023, Theorem 3]
- Theorem 6 covers a class of TDS that cannot be covered by [Chaillet et al., 2017, Theorem 8].
- The proof of Theorem 6 is based on [Karafyllis and Jiang, 2011, Lemma 6.7]

Conclusion

1. The principle of adding exponential term in relaxed LKFs to make them strict, does not systematically work even for $1D$ TDS.
2. Growth condition is proposed to conclude ISS with relaxed LKF.
3. The proposed condition turns out to extend the existing ones.
4. The conjecture is still an open question.

Discussion in the break!

For those who are intersted, I also get some other works:

1. E. Loko, A. Chaillet, I. Karafyllis (2024).
   Bulding coercive Lyapunov-Krasovskii functional based on Razumikhin and Halanay approaches.
   International Journal of Robust and Nonlinear Control.
2. E. Loko, A. Hayat (2024).
   Fredholm backstepping for general spectral operators.
   In process

Thank you for your attention!

Chaillet, A., Karafyllis, I., Pepe, P., and Wang, Y. (2023).
Growth conditions for global exponential stability and exp-ISS of time-delay systems under point-wise dissipation.
Systems & Control Letters, 178:105570.

Chaillet, A., Pepe, P., Mason, P., and Chitour, Y. (2017).
Is a point-wise dissipation rate enough to show ISS for time-delay systems?
IFAC-PapersOnLine, 50(1):14356–14361.

Karafyllis, I. and Jiang, Z.-P. (2011).
Stability and stabilization of nonlinear systems.
Springer Science & Business Media.

Karafyllis, I., Pepe, P., and Jiang, Z.-P. (2008).
Input-to-output stability for systems described by retarded functional differential equations.
European Journal of Control, 14(6):539–555.

Loko, E., Chaillet, A., Wang, Y., Karafyllis, I., and Pepe, P. (2024).
Growth conditions to ensure input-to-state stability of time-delay systems under point-wise dissipation. In process.

Orłowski, J., Chaillet, A., Destexhe, A., and Sigalotti, M. (2022).

Adaptive control of Lipschitz time-delay systems by sigma modification with application to neuronal population dynamics.
Systems & Control Letters, 159:105082.