A moment approach for entropy solutions of parameter-dependent hyperbolic conservation laws

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Parameter-dependant conservation laws

Consider the problem

$$\partial_t u(t,z,\xi) + \mathrm{div}_z f(u(t,z,\xi),\xi) = 0,\ (t,z,\xi) \in \mathbb{R}_+ \times \mathbb{R}^N \times \Xi,$$
$$u(0,z,\xi) = u_0(z,\xi),\ (z,\xi) \in \mathbb{R}^N \times \Xi,$$

where Ξ is a compact subset of $\mathbb{R}^p$ for some $p \in \mathbb{N}$.
We can rewrite this problem in the following form:

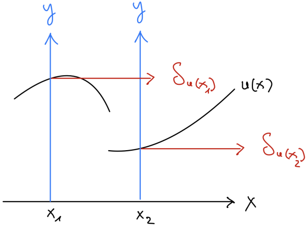
$$\nabla_x \cdot F(u(x,\xi);\xi) = 0,\ x \in D := \mathbb{R}_+ \times \mathbb{R}^N \ + \ \text{b.c.}$$

where $u(\cdot,\xi)$ is in a space V of functions defined on the domain $D \subset \mathbb{R}^n$ of "physical" (time and space) variables, by denoting $x = (t,z)$ and $F(u) = (u, f(u))$

Measure-theoretic approach to conservation laws

We adopt a measure-theoretic point of view of conservation laws, by considering measure-valued solutions [DiPerna 1985][1].
To a classical solution $u(x)$ corresponds a measure-valued solution (a map from D to the set of measure on $\mathbb{R}$)

$$\mu_x(dy) = \delta_{u(x)}(dy)$$



and the associated occupation measure ν on $D \times \mathbb{R}$ supported on the graph of u

$$\nu(dx, dy) = \mu_x(dy)dx.$$

[1] DiPerna, R.J.: Measure-valued solutions to conservation laws. Archive for Rational Mechanics and Analysis (1985)

Measure-theoretic approach to conservation laws

Consider the problem

$$\nabla_x \cdot F(u(x)) = 0, \; x \in D \subset \mathbb{R}^n$$

with boundary conditions $u = u_\Gamma$ on $\Gamma \subset \partial D$.

- It admits a weak form

$$\int_D \nabla_x \phi(x) \cdot F(u(x))dx - \underbrace{\int_\Gamma \phi(x) n \cdot F(u_\Gamma(x))dx}_{L_F(\phi)} = 0$$

  for some class of sufficiently smooth functions ϕ.

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- It even admits an ultra weak form on the measure-valued solution $\mu_x = \delta_{u(x)}$

$$\int_D \int \nabla_x \phi(x) \cdot F(y) \underbrace{\mu_x(dy)dx}_{\nu(dx,dy)} - L_F(\phi) = 0$$

  which is a linear equation in μ_x (or ν).

Measure-theoretic approach to conservation laws

In order to pick one of the solutions of the weak form, we impose additional conditions.

- Introduce a class $\mathcal{E}$ of "entropy" functions $E : \mathbb{R} \to \mathbb{R}^n$ such that

$$E'(u) = \eta'(u) F'(u)$$

for some sufficiently smooth and convex function η.

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- Consider the stronger form

$$\int \nabla_x \phi(x) \cdot E(u(x))dx - L_E(\phi) \geq 0, \ \forall E \in \mathcal{E},$$

and for all ϕ in some class of sufficiently smooth and positive test functions.

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- If $\mathcal{E}$ contains the functions $F(u)$ and $-F(u)$, this includes the previous weak conditions and introduce new ones.
- The solution is called the entropy solution for the class $\mathcal{E}$.

Measure-theoretic approach to conservation laws

The problem then admits an entropy measure-valued solution μ_x for the class $\mathcal{E}$, such that

$$\int\int \nabla_x \phi(x) \cdot E(y)\nu(dx, dy) - L_E(\phi) \geq 0, \ \forall E \in \mathcal{E},$$

and for all $\phi \geq 0$ in some class of smooth functions. Under some (smoothness) conditions, we prove that $\mu_x = \delta_{u(x)}$ with u the entropy solution.

Measure-theoretic approach to conservation laws

We end up with a linear formulation in the occupation measure ν

$$\int \nabla_x \phi(x) \cdot E(y)\ \nu(dx, dy) - L_E(\phi) \geq 0,\ \forall E \in \mathcal{E}$$

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- The occupation measure ν has for marginal in x the Lebesgue measure $\lambda(dx) = dx$. Restricting the problem to compact domains, this yields conditions

$$m_{\alpha,0}(\nu) = m_\alpha(\lambda)\ \forall \alpha.$$

Measure-theoretic approach to conservation laws

- In a polynomial setting, the problem is recasted as a moment problem

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- We then use a moment-SOS hierarchy that consists in minimizing over a truncated moment sequence $y^{(r)} = (y_\alpha)_{|\alpha| \leq 2r}$, $r \in \mathbb{N}$.
- To favorize approximate measures $\nu^{(r)}$ that are concentrated, we can choose (for a relaxation order r)

$$G(m(\nu)) = \mathrm{Tr}(M_r(\nu)),\ M_r(m(\nu)) = \int \phi_r(x,y)\phi_r(x,y)^T \nu(dx,dy),$$

where $\phi_r(x) = (x^\alpha)_{|\alpha| \leq r}$ is a basis of the space of polynomials of degree $\leq r$, which is a convex relaxation of a minimization of the rank of the moment matrix $M_r(m(\nu))$.

Moment methods to parameter-dependent conservation laws

The approach can be extended to parameter-dependent equations

$$\nabla_x \cdot F(u(x,\xi);\xi) = 0,\ x \in D \subset \mathbb{R}^n \ + \ \text{b.c.}$$

- Reformulation of the problem in terms of the occupation measure ν over $D \times \Xi \times \mathbb{R}$ supported on the graph of the solution $u(x,\xi)$

$$\nu(dx, d\xi, dy) = dx\rho(d\xi)\delta_{u(x,\xi)}(dy)$$

  with ρ some measure on the parameter space Ξ.
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  with ρ some measure on the parameter space Ξ.
  Requires new notions of weak-parametric entropy (measure-valued) solutions.
- Then apply a SOS hierarchy to get approximate moments of ν.

Moment methods to parameter-dependent conservation laws

Given approximate moments, different post-processing are possible.

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Given approximate moments, different post-processing are possible.

- Directly estimate quantities of interest

$$\int g(x,\xi,u(x,\xi))dx\rho(d\xi) = \int g(x,\xi,y)\nu(dx,d\xi,dy)$$

where g is a polynomial or well approximated by polynomials.

Moment methods to parameter-dependent conservation laws

Given approximate moments, different post-processing are possible.

- Recover the graph of the solution using Christoffel-Darboux approximation method [Marx et al. 2021][2].
  If $M_r(m(\nu))$ is invertible, the Christoffel-Darboux kernel is defined by

$$\kappa_{\nu,r}(a,b) = \phi_r(a)^T M_r(\nu)^{-1}\phi_r(b).$$

  Exploit the fact that the Christoffel-Darboux polynomial $\kappa_{\nu,r}((x,\xi,y),(x,\xi,y))$ tends to localize on the graph of u and approximate

$$u(x,\xi) \approx \underset{y}{\operatorname{argmin}}\, \kappa_{\nu,r}((x,\xi,y),(x,\xi,y))$$

[2] Marx, S., Pauwels, E., Weisser, T., Henrion, D., Lasserre, J.B.: Semi-algebraic approximation using Christoffel-Darboux kernel. Constructive Approximation (2021)

Illustration : Burgers equation

Consider the solution $u(t, x, \xi)$ of the Riemann problem for the Burgers equation

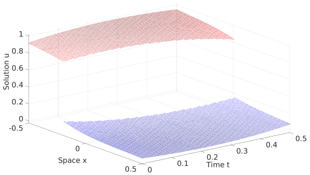
$$\partial_t u + \partial_x(u^2/2) = 0, \ t \in [0, 1/2], \ x \in [-1/2, 1/2] \qquad (\star)$$

with parametrized initial condition

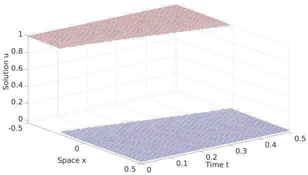
$$u_0(x, \xi) = \begin{cases} 1 \text{ if } x < \frac{1}{4}(\xi - 1), \\ 0 \text{ if } x \geq \frac{1}{4}(\xi - 1). \end{cases}$$

With $(t, x) = \mathbf{x}$ and $F(u) = (u, u^2/2)$, $(\star)$ writes

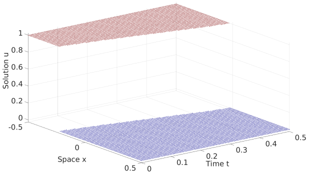
$$\nabla_{\mathbf{x}} F(u(\mathbf{x}, \xi)) = 0.$$



(a) $r = 2$

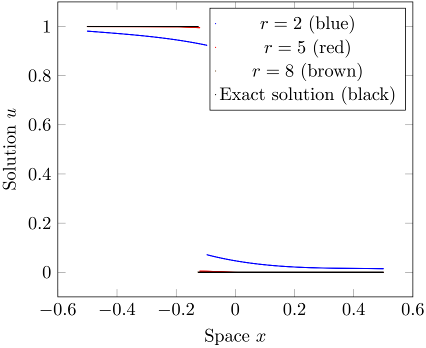


(b) $r = 5$



(c) $r = 8$

Figure: Graphs of the approximate solution for different relaxation orders r and $\xi = 0$



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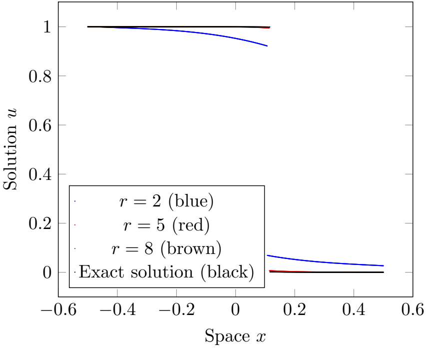

(b) $\xi = 1$

Figure: Graphs of the approximate solution for different relaxation orders and exact solution, at fixed ξ and time $t = 1/4$.

Thank you for your attention

Cardoen C., Marx S., Nouy A. and Seguin N.
A moment approach for entropy solutions of parameter-dependent hyperbolic conservation laws.
arXiv:2307.10043, 2023.