Low to high order finite element resolution for elliptic problems in the presence of a Dirac source term

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In a previous work [4], we conducted reduced order modeling of heat transfer within the human eyeball, focusing on studying the sensitivity of temperature outputs to model parameters. To obtain fast quadratic convergence in the output, we rely on a dual problem formulation associated with each output which is a linear functional of the solution. However, these output functionals, ℓ, can be pointwise evaluations of the solution, *e.g.* $\ell_x(v) = v(x) = \delta_x(v)$, for $x \in \Omega$. Thus, the formulation leads to a dual problem with Dirac source term δ_x with $x \in \Omega$ the point where we evaluate the solution of the primal problem. This is a specific application, but similar problems occur in a broad applicative context (*e.g.* accoustics, elasticity, *etc.*).
Theoretical and numerical aspects of Laplacian problem with Dirac source term and homogeneous Dirichlet boundary condition have been studied in [2, 1], *i.e.* given $x \in \Omega$, find u such that

$$\begin{aligned} -\Delta u &= \delta_x \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

[2, 1] show that the error of the finite element method converges with the same order as for standard problems, in Ω excluding a small region around the singularity.
In the present work, we propose a theorical review and a numerical exploration of these models, under various boundary conditions, and consider generalized elliptic problems, with discontinuous coefficients. We also investigate the impact of the position of the Dirac source term with respect to the border of the domain Ω. We discuss the results of our simulations with the finite element method from low to high order of discretization, obtained with the open-source software Feel++ [3].

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[2] T. Köppl, B. Wohlmuth. *Optimal A Priori Error Estimates for an Elliptic Problem with Dirac Right-Hand Side.* SIAM Journal on Numerical Analysis, **52(4)**, 1753–1769, 2014. doi : 10.1137/130927619.

[3] C. Prud'homme, V. Chabannes, S. Veys, T. Metivet, J.-B. Wahl, R. Hild, C. Daversin-Catty, T. Saigre, L. Sala, C. Trophime, L. Berti, A. Samaké, D. Prada, C. Van Landeghem, P. Jolivet. *feelpp/feelpp : Feel++ release v111 preview.9*, 2024. doi :10.5281/zenodo.10837178.

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