HMM schemes for fast-slows SDEs

Charles-Édouard Bréhier   Ludovic Goudenège
Jules Pertinand*

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Outline

Averaging of fast-slow systems

$$\begin{cases} \mathrm{d}X_t^\varepsilon &= f(X_t^\varepsilon, Y_t^\varepsilon)\,\mathrm{d}t + g(X_t^\varepsilon, Y_t^\varepsilon)\,\mathrm{d}B_t & \text{in } \mathbb{R}^d \\ \mathrm{d}Y_t^\varepsilon &= \frac{1}{\varepsilon} b(X_t^\varepsilon, Y_t^\varepsilon)\,\mathrm{d}t + \frac{1}{\sqrt{\varepsilon}}\sigma(X_t^\varepsilon, Y_t^\varepsilon)\,\mathrm{d}W_t & \text{in } \mathbb{R}^d \end{cases}$$

- Dynamic frozen in x: $\mathrm{d}Y_t^x = b(x, Y_t^x)\,\mathrm{d}t + \sigma(x, Y_t^x)\,\mathrm{d}W_t$
  Assume
  $$Y_t^x \xrightarrow[t\to\infty]{law} Y_\infty^x \sim \mu^x$$
- *Averaging* : for all $T < \infty$,
  $$\operatorname{dist}(law(X_T^\varepsilon), law(\overline{X_T})) = O(\varepsilon)$$
  where
  $$\mathrm{d}\overline{X}_t = \overline{f}(\overline{X}_t)\,\mathrm{d}t + \overline{g}(\overline{X}_t)\,\mathrm{d}\overline{B}_t$$
  and
  $$\overline{f}(x) = \int_{\mathbb{R}^d} f(x,y)\,\mathrm{d}\mu^x(y), \quad \overline{gg}^T(x) = \int_{\mathbb{R}^d} gg^T(x,y)\,\mathrm{d}\mu^x(y).$$

Heterogeneous Multiscale Method schemes

E-Liu-Vanden-Eijnden 2005 : Approximate $\overline{X}_T$!

- Macro-scheme : $\Delta t > 0$

$$X_{n+1} = X_n + \Delta t F(X_n) + \Delta t^{\frac{1}{2}} G(X_n) \Gamma_{n+1}$$

with $(F, G) \simeq (\overline{f}, \overline{g})$

- Micro-scheme : $\tau > 0$

$$Y^x_{n+1} = \phi(x, Y^x_n, \tau, \gamma_{n+1})$$

s.t.

$$Y^x_n \xrightarrow[n\to\infty]{law} Y^x_\infty \sim \mu^x_\tau \simeq \mu^x$$

- Ergodic averages

$$F(x) = \frac{1}{K} \sum_{k=k_b}^{K+k_b} f(x, Y^x_k), \quad GG^T(x) = \frac{1}{K} \sum_{k=k_b}^{K+k_b} gg^T(x, Y^x_n)$$

$$(F, G) \xrightarrow[K\to\infty]{} (\overline{f}, \overline{g})$$

Choice of τ, K, k_b ?

Our scheme

$$\begin{cases} X^\tau_{n+1} &= X^\tau_n + \Delta t f(X^\tau_n, Y^\tau_{n+1}) + \Delta t^{\frac{1}{2}} g(X^\tau_n, Y^\tau_{n+1})\Gamma_{n+1} \\ Y^\tau_{n+1} &= Y^\tau_n + \tau b(X^\tau_n, Y^\tau_{n+1}) + \tau^{\frac{1}{2}} \sigma(X^\tau_n, Y^\tau_n)\gamma_{n+1} \end{cases}$$

Theorem

Let $T > 0$, $\Delta t, \tau > 0$. For $N = \frac{T}{\Delta t}$ and a test function φ we have

$$\left|\mathbb{E}\left[\varphi(X^\tau_N)\right] - \mathbb{E}\left[\varphi(\overline{X}_T)\right]\right| \lesssim \Delta t + \tau + \frac{\Delta t}{\tau}$$

assuming

- φ *and* f, g, b, σ *smooth*
- b *strictly dissipative :* $\exists L > 0$ *s.t.* $\forall x, y$

$$(b(x) - b(y), x - y) \leq -L|x - y|^2$$

$$\forall h, \quad (\nabla b(x)h, h) \leq -Lh^2$$

$$\forall h, \quad (\nabla^2 V h, h) \geq Lh^2 \quad \textit{if} \quad b = -\nabla V$$

Cost analysis

- Reminder : $\left|\mathbb{E}\left[\varphi(X_T^\varepsilon)\right] - \mathbb{E}\left[\varphi(\overline{X}_T)\right]\right| \lesssim \varepsilon$

Corollary

Under the same assumptions, $\tau = \Delta t^{\frac{1}{2}}$ is an optimal choice and

$$\left|\mathbb{E}\left[\varphi(X_N^\tau)\right] - \mathbb{E}\left[\varphi(X_T^\varepsilon)\right]\right| \lesssim \varepsilon + \Delta t^{\frac{1}{2}}$$

- $\text{Cost} = N_{\text{macrosteps}} \times N_{\text{microsteps per macrosteps}} \propto \frac{1}{\Delta t}$
- AP schemes : Bréhier-Rakotonirina-Ricquebourg 2020

$$\sup_{\varepsilon \in (0,1)} \left|\mathbb{E}\left[\varphi(X_T^\varepsilon)\right] - \mathbb{E}\left[\varphi(\tilde{X}_N)\right]\right| \lesssim \Delta t^{\frac{1}{2}}$$

Limited to Ornstein-Uhlenbeck fast dynamic !

Simulation : test case

- Ornstein-Uhlenbeck fast dynamic
  $b(x,y) = -(y-x)$, $\sigma(x,y) = \cos(x)$

$$\mu^x = \mathcal{N}\left(x, \frac{\cos^2(x)}{2}\right)$$

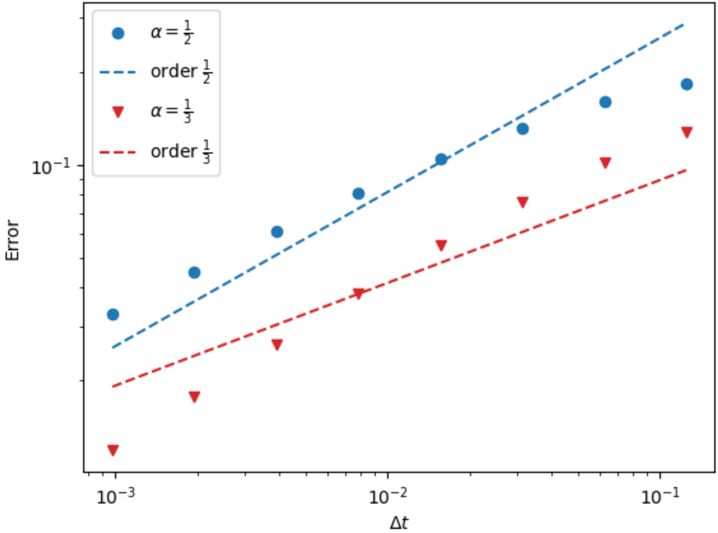
- Modulated slow dynamic
  $f(x,y) = \cos(y)x + \sin(y+\frac{1}{2})\arctan(x+1)$

$$\overline{f}(x) = \int_{\mathbb{R}^d} \cos(y)\,\mathrm{d}\mu^x(y)x + \int_{\mathbb{R}^d} \sin(y+\tfrac{1}{2})\,\mathrm{d}\mu^x(y)\arctan(x+1)$$

$$g(x,y) = 0 \Rightarrow \overline{g}(x) = 0$$

- $\tau = \Delta t^\alpha \rightsquigarrow \mathrm{Error} \lesssim \Delta t + \Delta t^\alpha + \Delta t^{1-\alpha}$

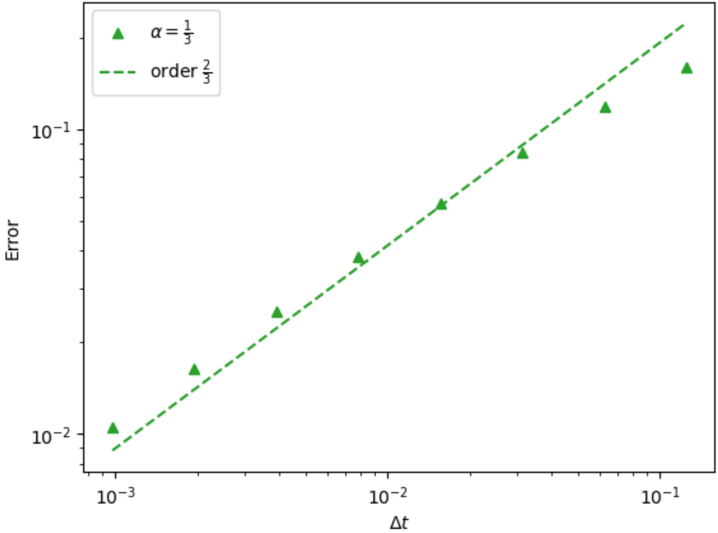
Order of convergence



$$\text{Error} \lesssim \Delta t + \Delta t^{\alpha} + \Delta t^{1-\alpha}$$

$M = 10^5 - 10^7$

Order of convergence : Exponential Euler



$$\text{Error} \lesssim \Delta t + \Delta t^{1-\alpha}$$

Main tools

- Proof : $\left|\mathbb{E}\left[\varphi(X_T^\varepsilon)\right] - \mathbb{E}\left[\varphi(\overline{X}_T)\right]\right| \lesssim \varepsilon$
- Adapt strategy to discrete time $\varepsilon \rightsquigarrow \frac{\Delta t}{\tau}$
- Kolmogorov equation

$$\mathbb{E}\left[\varphi(X_T^\varepsilon)\right] - \mathbb{E}\left[\varphi(\overline{X}_T)\right] = \int_0^T \left(\mathcal{L} - \overline{\mathcal{L}}\right) \overline{u}(T-t, X_t^\varepsilon)\,\mathrm{d}t$$

- Poisson equation

$$-\mathcal{L}_Y \psi[h] = h - \overline{h}$$

- Regularity theory

$$\|\psi\|_{C^2} + \|\overline{u}\|_{C^2} \lesssim 1$$

Kolmogorov equation

- $(X^\varepsilon, Y^\varepsilon) \rightsquigarrow \mathcal{L} = \mathcal{L}_X + \frac{1}{\varepsilon}\mathcal{L}_Y$ with

$$\mathcal{L}_X = f(x,y)\cdot\nabla_x + \frac{1}{2}gg^T(x,y) : \nabla_x^2$$

$$\mathcal{L}_Y = b(x,y)\cdot\nabla_y + \frac{1}{2}\sigma\sigma^T(x,y) : \nabla_y^2$$

- $\overline{X} \rightsquigarrow \overline{\mathcal{L}} = \overline{f}(x)\cdot\nabla_x + \frac{1}{2}\overline{gg}^T(x) : \nabla_x^2$
- $\overline{u}(t,x) = \mathbb{E}\left[\varphi(\overline{X}_t) \mid \overline{X}_0 = x\right] \rightsquigarrow \begin{cases} \partial_t\overline{u} - \overline{\mathcal{L}}\overline{u} & = & 0 \\ \overline{u}(t=0) & = & \varphi \end{cases}$

$$\begin{aligned}
\mathcal{E} &= \mathbb{E}\left[\varphi(X_T^\varepsilon)\right] - \mathbb{E}\left[\varphi(\overline{X}_T)\right] = \mathbb{E}\left[\overline{u}(0, X_T^\varepsilon)\right] - \overline{u}(T, x = X_0^\varepsilon) \\
&= \mathbb{E}\int_0^T (\mathcal{L}_X + \frac{1}{\varepsilon}\mathcal{L}_Y - \overline{\mathcal{L}})\overline{u}(T-t, X_t^\varepsilon)\,\mathrm{d}t \\
&= \mathbb{E}\int_0^T \Big((f-\overline{f})\cdot\nabla_x + \frac{1}{2}(gg^T - \overline{gg}^T) : \nabla_x^2\Big)\overline{u}(T-t, X_t^\varepsilon)\,\mathrm{d}t \\
&= \mathbb{E}\int_0^T h_t(X_t^\varepsilon, Y_t^\varepsilon) - \overline{h_t}(X_t^\varepsilon)\,\mathrm{d}t
\end{aligned}$$

Poisson equation

- $h(x,y) \rightsquigarrow \psi[h](x,y)$:

$$-\mathcal{L}_Y \psi[h](x,y) = h(x,y) - \overline{h}(x)$$

$\mathcal{L}_Y = b(x,y) \cdot \nabla_y + \frac{1}{2}\sigma\sigma^T(x,y) : \nabla_y^2$

- Reformulation of the remainder :

$$\mathbb{E}\left[\psi(X_T^\varepsilon, Y_T^\varepsilon)\right] - \mathbb{E}\left[\psi(x,y)\right] = \mathbb{E}\int_0^T (\mathcal{L}_X + \frac{1}{\varepsilon}\mathcal{L}_Y)\psi(X_t^\varepsilon, Y_t^\varepsilon)\,\mathrm{d}t$$

$$\begin{aligned}\Rightarrow \mathbb{E}\int_0^T h(X_t^\varepsilon, Y_t^\varepsilon) - \overline{h}(X_t^\varepsilon)\,\mathrm{d}t &= \varepsilon\Big(\mathbb{E}\int_0^T \mathcal{L}_X\psi(X_t^\varepsilon, Y_t^\varepsilon)\,\mathrm{d}t \\ &\qquad - \mathbb{E}\left[\psi(X_T^\varepsilon, Y_T^\varepsilon)\right] + \mathbb{E}\left[\psi(x,y)\right]\Big) \\ &= O(\varepsilon)\end{aligned}$$

- Need to bound

$$\|\overline{u}\|_{C^2} \rightsquigarrow \|(\overline{f}, \overline{g})\|_{C^2} \quad \text{and} \quad \|\psi\|_{C^2}$$

Regularity of the invariant measure

- Implicit dependance in x

$$\nabla_x \overline{f} = \nabla_x \int f(x,y)\mu^x(\,\mathrm{d}y)$$

- Use the PDE : $-\mathcal{L}_Y^x \psi(x,y) = f(x,y) - \overline{f}(x)$

$$-\mathcal{L}_Y^x \nabla_x \psi = \nabla_x f - \nabla_x \overline{f} + \left(\nabla_x \mathcal{L}_Y^x\right)\psi$$

- $\mathcal{L}^{x*}_Y \mu^x = 0 \Rightarrow \int \mathcal{L}_Y^x \psi(x,y)\mu^x(\,\mathrm{d}y) = 0$

$$\nabla_x \overline{f} = \int \nabla_x f(x,y) + \left(\nabla_x \mathcal{L}_Y^x\right)\psi^x \mu^x(\,\mathrm{d}y)$$

- $\nabla_x \mathcal{L}_Y^x = \nabla_x b(x,y) \cdot \nabla_y + \frac{1}{2}\nabla_x \left(\sigma\sigma^T\right)(x,y) : \nabla_y^2$

Need to control $\nabla_y^k \psi$!

Regularity of the corrector

- $Y^x(y) \rightsquigarrow \mathcal{L}_Y^x = b(x,y)\cdot\nabla_y + \frac{1}{2}\sigma\sigma^T(x,y) : \nabla_y^2$
- Representation formula : $-\mathcal{L}_Y\psi[h] = h - \overline{h}$

$$\begin{aligned}\psi[h](x,y) &= \int_0^\infty e^{-t\mathcal{L}_Y}\left(h-\overline{h}\right)(y)\,\mathrm{d}t \\ &= \int_0^\infty \mathbb{E}\left[h(x,Y_t^x(y)) - \overline{h}(x)\right]\,\mathrm{d}t\end{aligned}$$

$$\Rightarrow \nabla_y\psi[h] = \int_0^\infty \mathbb{E}\left[\nabla_y h(Y_t^x)\cdot\nabla_y Y_t^x(y)\right]\,\mathrm{d}t$$

- Strict dissipativity of b

$$\mathbb{E}\left[|Y_t^x(y_1) - Y_t^x(y_2)|^2\right]^{\frac{1}{2}} \lesssim e^{-(L-\frac{1}{2}\|\nabla\sigma\|_\infty^2)t}|y_1-y_2|$$

$$\mathbb{E}\left[|\nabla_y Y^x(y)|^2\right]^{\frac{1}{2}} \lesssim e^{-(L-\frac{1}{2}\|\nabla\sigma\|_\infty^2)t}$$

Summary

- Goal

$$\text{Approximate } \mathbb{E}\left[\varphi(X_T^\varepsilon)\right]$$

- Averaging

$$X^\varepsilon \simeq \overline{X} + O(\varepsilon)$$

- 1-step HMM scheme

$$\overline{X}_T \simeq X_N^\tau + O(\Delta t^{\frac{1}{2}})$$

Beyond ?

- Diffusion approximation : $X^{\varepsilon}_{\frac{T}{\varepsilon}}$
- Variance estimate $\Rightarrow$ Central Limit Theorem for X^{τ}_{N}
- Dissipativity assumption

$$(b(x), x) \leq -L|x|^2 + C$$

- Fractionnal SDEs
- SPDEs

Thanks for your attention !