# A depth-averaged heart model for the inverse problem of cardiac electrophysiology

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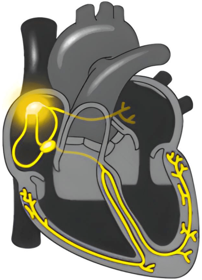
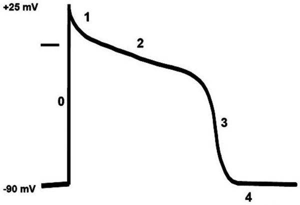
- **Context**
- Forward averaged model
- Inverse problem
- Numerical results

# Heart electrophysiology



- Cardiac rhythm disorders: pathologies of heart's electrical activity
- Electrical activation: gives the signal that precedes the contraction



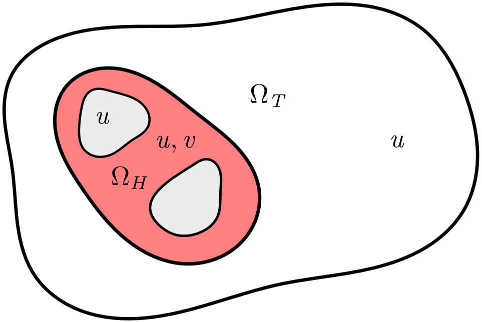

- Action potential
- Activation maps
- Goal: detect pathologies from torso potentials, ECG imaging. Very ill posed inverse problem 😿

# Bidomain model



Potential $u \in H^1(\Omega_T \cup \Omega_H)$, voltage $v \in H^1(\Omega_H)$.

$$
\begin{aligned}
&\mathrm{div}(\sigma_i \nabla (u+v)) = \partial_t v + I_{ion}(v,t,h) && \Omega_H\\
&\mathrm{div}((\sigma_e + \sigma_i)\nabla u) = -\mathrm{div}(\sigma_i \nabla v) && \Omega_H\\
&\mathrm{div}(\sigma_T \nabla u) = 0 && \Omega_T\\
&\sigma_i \nabla (u+v)\cdot n = 0 && \partial\Omega_H\\
&\sigma_T \nabla u \cdot n = 0 && \partial\Omega_T
\end{aligned}
$$

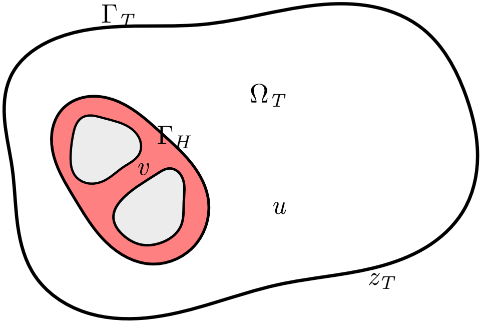


# The inverse problem of cardiac electrophysiology

Find $u$ and/or $v$ that satisfy (part of) the bidomain equations, matching torso data $z_T$.
*Find the activation map.*

$$\begin{aligned}
&\mathrm{div}((\sigma_e+\sigma_i)\nabla u) = -\mathrm{div}(\sigma_i \nabla v) && \Omega_H\\
&\mathrm{div}(\sigma_T \nabla u) = 0 && \Omega_T\\
&\sigma_i \nabla(u+v)\cdot n = 0 && \partial\Omega_H\\
&\sigma_T \nabla u \cdot n = 0 && \partial\Omega_T
\end{aligned}$$

- In the heart surface: $u$: uniqueness (up to a constant) but non existence, $v$ can be found up to a constant
- In the heart volume: infinite-dimensional kernel for $v$

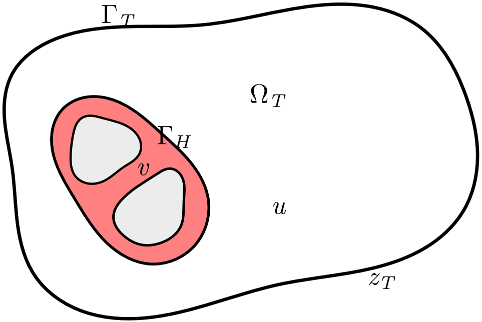


# The usual (static) potential inverse problem

Cauchy problem:

$$\begin{array}{ll} -\Delta u = 0 & \Omega_T \\ u = z_T & \Gamma_T \\ \partial_n u = 0 & \Gamma_T \end{array}$$

- No interactions with the heart
- Does not allow to reconstruct $v$
- No information on the heart volume



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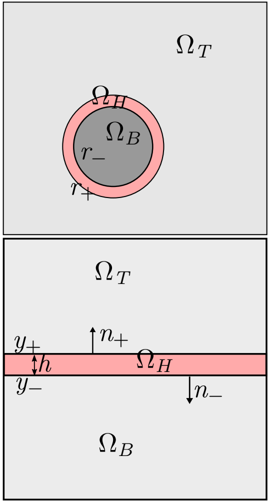
# Assumptions

- 2D + smooth
- $\Omega_H$ separates blood cavity from torso
- Depth $h \ll L$, $h'(x) \ll 1$
- $\Gamma_H$ smoothe curve embedded in $\Omega_H$
- Parallel fibers:

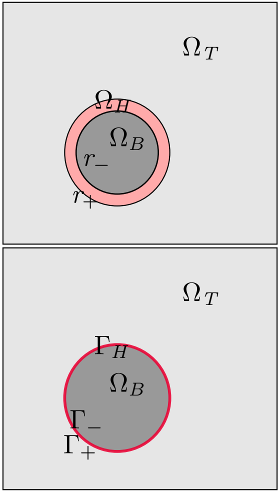
$$\sigma_{i/e} = \begin{pmatrix} \sigma_{i/e,\ell} & 0 \\ 0 & \sigma_{i/e,p} \end{pmatrix}$$



$$\begin{array}{ll} \sigma_\ell h \frac{\partial^2}{\partial s^2} \bar{u} + \sigma_T \frac{\partial u_T}{\partial n}|_{\Gamma_H} + \sigma_B \frac{\partial u_B}{\partial n}|_{\Gamma_H} + \sigma_{i\ell} h \frac{\partial^2}{\partial s^2} \bar{v} = 0 & \Gamma_H \\ \mathrm{div}(\sigma_T \nabla u_T) = 0 & \Omega_T \\ \mathrm{div}(\sigma_B \nabla u_B) = 0 & \Omega_B \end{array}$$

+BC:

$$\begin{array}{ll} \sigma_T \partial_n u = 0 & \Gamma_T \\ \sigma_{ep} \frac{u_T - \bar{u}}{h} = (1-\alpha)\sigma_T \partial_n u_T & \Gamma_H \\ \sigma_{ep} \frac{u_B - \bar{u}}{h} = \alpha \sigma_B \partial_n u_B & \Gamma_H \end{array}$$



# Variational formulation



Find $(\bar{u}, u_T, u_B)$ in $V = H^1(\Gamma_H) \times H^1(\Omega_T)/\mathbb{R} \times H^1(\Omega_B)$ such that

$$\boxed{\begin{aligned}&\sigma_\ell h \int_{\Gamma_H} \frac{\partial}{\partial s}\bar{u}\frac{\partial}{\partial s}\bar{\varphi} + \sigma_T \int_{\Omega_T} \nabla u_T \cdot \nabla \varphi_T + \sigma_B \int_{\Omega_T} \nabla u_B \cdot \nabla \varphi_B \\ &+ \frac{\sigma_{e,p}}{(1-\alpha)h} \int_{\Gamma_H} (\bar{u} - u_T)(\bar{\varphi} - \varphi_T) + \frac{\sigma_{e,p}}{\alpha h} \int_{\Gamma_H} (\bar{u} - u_B)(\bar{\varphi} - \varphi_B) = \int_{\Gamma_H} F\bar{\varphi}, \quad \forall(\bar{\varphi}, \varphi_T, \varphi_B) \in V\end{aligned}}$$

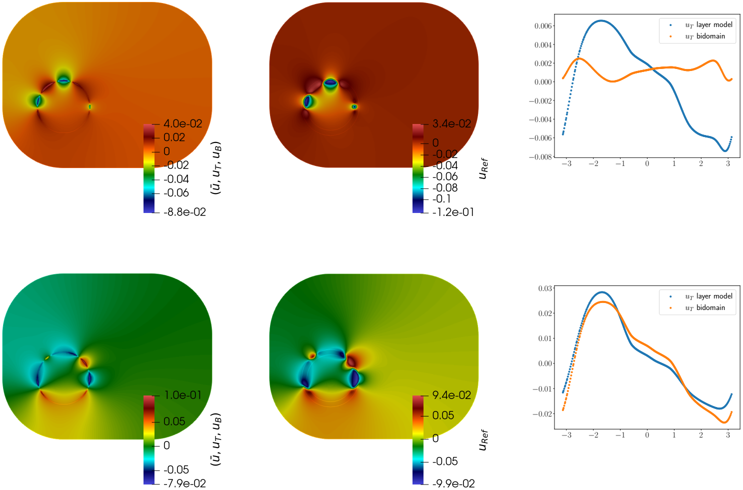
$F = h\sigma_{i,\ell}\partial_{ss}\bar{v}$

***Well posed, all Lax-Milgram hypotheses are satisfied !***

# Numerical validation

## Comparison with the bidomain





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# Optimal control



$$\min_{\mathcal{E}} J(U[F], F) = \frac{1}{2} \int_{\Gamma_T} |u_T - z_T|^2 + \frac{\varepsilon_{\text{regul}}}{2} |F|^2_{H^1(\Gamma_H)}$$

Over space $\mathcal{E} = \left\{(U, F) \,|\, U = (u_T, u_B, \bar{u}) = AF + \mathbb{R},\ F \in H^1(\Gamma_H)/\mathbb{R}\right\}$ with operator $A$:

$$\begin{aligned} &\sigma_\ell h \int_{\Gamma_H} \frac{\partial}{\partial s}\bar{u}\frac{\partial}{\partial s}\bar{\varphi} + \sigma_T \int_{\Omega_T} \nabla u_T \cdot \nabla \varphi_T + \sigma_B \int_{\Omega_T} \nabla u_B \cdot \nabla \varphi_B \\ &+ \frac{\sigma_{e,p}}{(1-\alpha)h} \int_{\Gamma_H} (\bar{u} - u_T)(\bar{\varphi} - \varphi_T) + \frac{\sigma_{e,p}}{\alpha h} \int_{\Gamma_H} (\bar{u} - u_B)(\bar{\varphi} - \varphi_B) = \int_{\Gamma_H} F\bar{\varphi} \end{aligned}$$

# Existence & uniqueness of the optimal control

With regularization



Steps of the proof:

- Existence:
    1. **Minimizing sequence**
    2. **Weakly convergent subsequences**
    3. **Use of lower semi continuity**
- Uniqueness: Strict convexity

$$J(U, F) = \frac{1}{2} \int_{\Gamma_T} |u_T - z_T|^2 + \frac{\varepsilon_{\text{regul}}}{2} |F|^2_{H^1}$$

# Lagrangian formulation



$$\begin{aligned}\mathcal{L}(\bar{u}, u_T, u_B, \bar{\lambda}, \lambda_T, \lambda_B, F) = {} & \frac{1}{2}\int_{\Gamma_T} |u_T - z_T|^2 + \frac{\varepsilon_{\text{regul}}}{2}\int_{\Gamma_H} |\partial_s F|^2 \\ & + \sigma_\ell h \int_{\Gamma_H} \partial_s \bar{u} \partial_s \bar{\lambda} + \frac{\sigma_{e,p}}{(1-\alpha)h}\int_{\Gamma_H} (\bar{u} - u_T)(\bar{\lambda} - \lambda_T) \\ & + \frac{\sigma_{e,p}}{\alpha h}\int_{\Gamma_H} (\bar{u} - u_B)(\bar{\lambda} - \lambda_B) \\ & + \sigma_T \int_{\Omega_T} \nabla u_T \cdot \nabla \lambda_T + \sigma_B \sigma_B \int_{\Omega_B} \nabla u_B \cdot \nabla \lambda_B - \int_{\Gamma_H} F\bar{\lambda}\end{aligned}$$

**1st order optimality conditions ➔ *linear* system to solve 😁**

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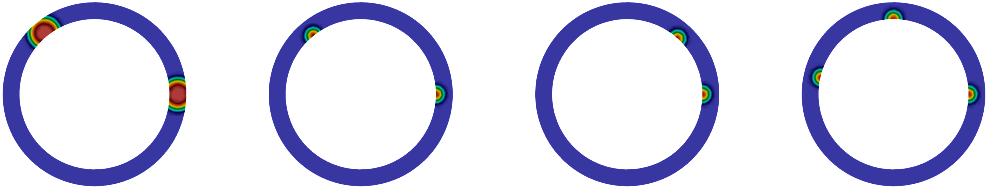
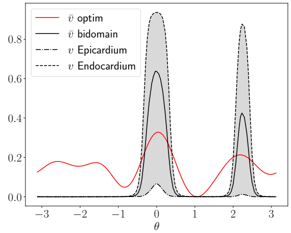
# Protocol

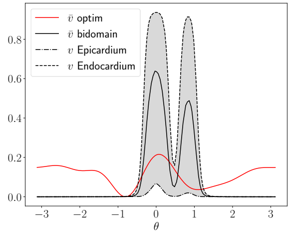




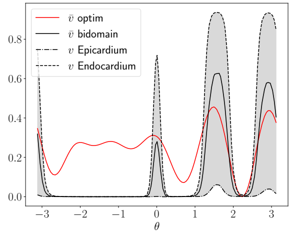
(a) Pacing protocol 1 (b) Pacing protocol 2 (c) Pacing protocol 3 (d) Pacing protocol 4



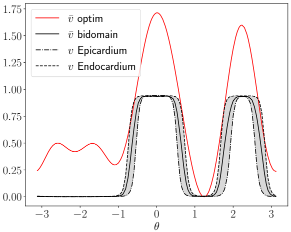
(a) Protocol 2. t=6 ms



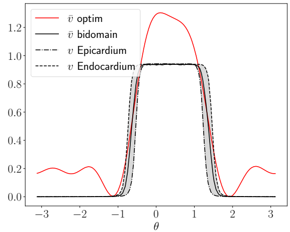
(b) Protocol 3. t=6 ms



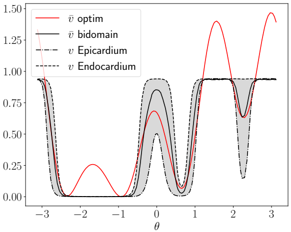
(c) Protocol 4. t=6 ms



(d) Protocol 2. t=13 ms



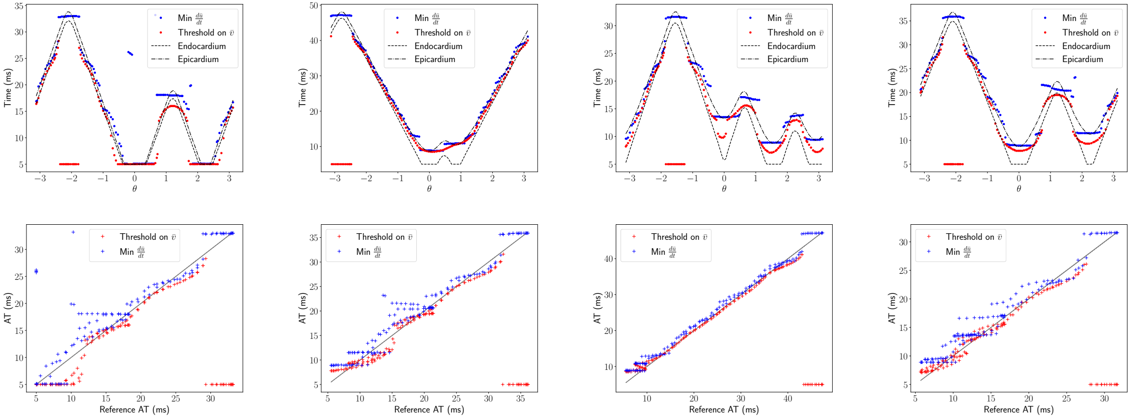
(e) Protocol 3. t=13 ms



(f) Protocol 4. t=13 ms

# Activation maps






(a) Protocol 1 (b) Protocol 2 (c) Protocol 3 (d) Protocol 4

# Conclusion



- Accurate activation maps, bounded by endocardial & epicardial ATs
- We recover an average $v$, some information on the volume
- Comparison with the usual epicardial reconstruction
- Perspective of 3D + more realistic geometry

**Thank you for your attention !**

Emma Lagracie, Yves Bourgault, Yves Coudière and Lisl Weynans, *A depth-averaged heart model for the inverse problem of cardiac electrophysiology*, Submitted

Operator $T$: $AF|\Gamma_T = ([\text{Trace}] \circ A)F = TF$ continuous Banach to Banach

- **Minimizing sequence:** $m = \inf_{\mathcal{E}} J(U, F) \geq 0$, minimizing sequence $(U_n, F_n) \in \mathcal{E}_2$ s.t. $J(U_n, F_n) \overset{n\to\infty}{\rightarrow} m$.
- **Weakly convergent subsequences**

$$\frac{1}{2}\int_{\Gamma_T} |u_{n_T} - z_T|^2 + \frac{\varepsilon_{\text{regul}}}{2} |F_n|^2_{H^1} \leq J(0,0) = Cte. \tag{1}$$

— $(F_n)$ bounded

— $(u_{nT|\Gamma_T}) = (TF_n + C_n)$ bounded + $(TF_n)$ bounded (continuity) $\Rightarrow$ $(C_n)$ bounded

Subsequences

— $(F_n) \underset{n\to\infty}{\rightharpoonup} \bar{F}$

— $(C_n) \overset{n\to\infty}{\rightarrow} \bar{C}$

— ($T$ weakly continuous) $TF_n \underset{n\to\infty}{\rightharpoonup} T\bar{F}$

- **use of lower semi continuity** $F \longmapsto \|F\|^2$ weakly l.s.c.

$$\frac{\varepsilon_{\text{regul}}}{2}\|\bar{F}\|^2 \leq \liminf_{n\to\infty} \frac{\varepsilon_{\text{regul}}}{2}\|F_n\|^2,$$

$$\frac{1}{2}\int_{\Gamma_T} \left|T\bar{F} + \bar{C} - z_T\right|^2 \leq \liminf_{n\to\infty} \frac{1}{2}\int_{\Gamma_T} \left|u_{n\,T} - z_T\right|^2$$

Finally,

$$J(\bar{U}, \bar{F}) \leq \liminf_{n\to\infty} J(U_n, F_n) = m.$$

**Uniqueness:** $J$ strictly convex.