Approximating a continuously stratified hydrostatic system by the multi-layer shallow water system

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Goal: Establishing a bridge between multilayer and continuously stratified hydrostatic flows.

Considered systems:

- The continuously stratified hydrostatic system (**CSHS**).
- The multi-layer shallow water system (**MSWS**).

(CSHS+GM)

$$\begin{cases} \partial_t h + \partial_x((\underline{h} + h)(\underline{u} + u)) = \kappa \partial_x^2 h \\ \partial_t u + \left(\underline{u} + u - \kappa \frac{\partial_x h}{\underline{h}+h}\right) \partial_x u + \frac{1}{\varrho} \mathscr{M}(\partial_x h) = 0 \end{cases}$$

Where

$$\mathscr{M}(\partial_x h(t,x))(\varrho) = \int_{\rho_{surf}}^{\rho_{bott}} \min(\rho, \rho') \partial_x h(t,x,\varrho') d\varrho'.$$

(MSWS+GM)

$$\begin{cases} \partial_t H_i + \partial_x((\underline{H}_i + H_i)(\underline{U}_i + U_i)) = \kappa \partial_x^2 H_i, \quad \forall i \in \{1, \cdots, N\} \\ \partial_t U_i + \left(\underline{U}_i + U_i - \kappa \frac{\partial_x H_i}{\underline{H}_i + H_i}\right) \partial_x U_i + g \sum_{j=1}^{N} \frac{1}{N} \frac{\min(\boldsymbol{\rho}_i, \boldsymbol{\rho}_j)}{\boldsymbol{\rho}_i} \partial_x H_j = 0. \end{cases}$$

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Where

- $t > 0, x \in \mathbb{R}$.
- $\underline{H}_i$, $\underline{U}_i$, $\boldsymbol{\rho}_i \in \mathbb{R}$.
- H_i, U_i are the deviation of the equilibrium $\underline{H}_i$, $\underline{U}_i$.
- g denotes the acceleration of gravity.
- The densities satisfy $\boldsymbol{\rho}_i = \rho_{surf} + \frac{i-1}{N}(\rho_{bott} - \rho_{surf})$ $\forall i \in \{1, \cdots, N\}$, with $\rho_{bott} > \rho_{surf} > 0$, $\rho_{bott} - \rho_{surf} = 1$.

Remarks

- Euler.eq+$\underbrace{\partial_z P + g\rho = 0}_{Hyd.app}$+Isopycnal coordinates $\Longrightarrow$ **(CSHS)**
- The κ terms are motivated by the work of the oceanographers Gent & McWilliams on isopycnal mixing and eddy diffusivity (in the 90's), and which could be interpreted as turbulence terms. Moreover the adding of this diffusive κ term in the first equation has a regularizing effect.
- The system (**CSHS**+**GM**) is well-posed in Sobolev spaces on the time interval $[0, T]$ with $T^{-1} = C(1 + \kappa^{-1}(|\underline{u}'|^2_{L^2_\varrho} + M_0^2))$, where M_0 is the size of the initial data, and C depends only on M_0 and the size of the equilibrium $(\underline{\rho}, \underline{u})$. [Duchêne & Bianchini '22].

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The continuous case	The discrete case
ϱ	$\boldsymbol{\rho} = (\boldsymbol{\rho}_1, \cdots, \boldsymbol{\rho}_N)^t$
h	$(H_i)_{i=1,\cdots,N} = (h(\boldsymbol{\rho}_i))_{i=1,\cdots,N}$
u	$(U_i)_{i=1,\cdots,N} = (u(\boldsymbol{\rho}_i))_{i=1,\cdots,N}$
$\int_{\varrho}^{\rho_{bott}} f \ d\varrho'$	$(\mathsf{S}F)_i = \sum_{j=i}^{N} (\boldsymbol{\rho}_j - \boldsymbol{\rho}_{j-1}) \ F_{j-1}$
$\partial_{\varrho} f$	$(\mathsf{D}_{\rho}F)_i = \frac{1}{\boldsymbol{\rho}_i - \boldsymbol{\rho}_{i+1}}(F_i - F_{i+1})$
$\mathscr{M} f(\varrho) = \int_{\rho_{surf}}^{\rho_{bott}} \min(\varrho, \rho') f(\varrho') d\varrho'$	$\sum_{j=1}^{N} \frac{\min(\boldsymbol{\rho}_i, \boldsymbol{\rho}_j)}{N} F_j$

Main result

(**MSWS+GM**)

$$\begin{cases} \partial_t H_i + \partial_x((\underline{H}_i + H_i)(\underline{U}_i + U_i)) = \kappa \partial_x^2 H_i, \quad \forall i \in \{1, \cdots, N\} \\ \partial_t U_i + \left(\underline{U}_i + U_i - \kappa \frac{\partial_x H_i}{\underline{H}_i + H_i}\right) \partial_x U_i + g \sum_{j=1}^{N} \frac{1}{N} \frac{\min(\rho_i, \rho_j)}{\rho_i} \partial_x H_j = 0. \end{cases}$$

[MA '23]

Let $s \in \mathbb{N}$ be such that $s > 2 + \frac{1}{2}$, there exists $C > 0$ such that for any $N \in \mathbb{N}^*$ and any $\kappa > 0$, for any initial data $(H_0, U_0) \in H^{s,2}$, satisfying natural assumptions and with $M_0 := |||(H_0, U_0)|||_s$, the following holds. Denoting

$$T^{-1} = C\left(1 + \kappa^{-1}\left(\| \mathrm{D}_\rho \underline{U} \|_{l^2}^2 + M_0^2\right)\right),$$

there exists a unique strong solution (H, U) to (**MSWS+GM**) with initial data $(H, U)|_{t=0} = (H_0, U_0)$. and one has, for any $t \in [0, T]$ the estimate $|||(H, U)(t, \cdot)|||_s \leqslant CM_0$.

Remarks

- We have

$$|||(H,U)|||_s = \sum_{j=0}^{1} \| \mathrm{D}_\rho^j H \|_{H_x^{s-j}} + \sum_{j=0}^{2} \| \mathrm{D}_\rho^j \mathrm{S} H \|_{H_x^{s-j}} + \sum_{j=0}^{2} \| \mathrm{D}_\rho^j U \|_{H_x^{s-j}} + \| \mathrm{TS} H \|_{H_x^s}.$$

- The time of existence is independent of the number of layers N.
- There is an obvious similarity between the time $T^{-1} = C\left(1 + \kappa^{-1}\left(\| \mathrm{D}_\rho \underline{U} \|_{l^2}^2 + M_0^2\right)\right)$ obtained in the previous Theorem and the one of the WP of (**CSHS+GM**).

(MSWS+GM)

$$\begin{cases} \partial_t H_i + \partial_x((\underline{H}_i + H_i)(\underline{U}_i + U_i)) = \kappa \partial_x^2 H_i, \quad \forall i \in \{1, \cdots, N\} \\ \partial_t U_i + \left(\underline{U}_i + U_i - \kappa \frac{\partial_x H_i}{\underline{H}_i + H_i}\right) \partial_x U_i + g \sum_{j=1}^{N} \frac{1}{N} \frac{\min(\rho_i, \rho_j)}{\rho_i} \partial_x H_j = 0. \end{cases}$$

Tools of the proof

- The small time existence and uniqueness part of the proof relies on existence and uniqueness theorems of transport and transport diffusion equations with their corresponding estimates.
- The long time existence and uniqueness is based on the energy method. Using our correspondence (Dictionary) the estimates derive naturally from the the estimates found in the continuous case [Duchêne & Bianchini '22].

Question: What happens when the number of layers N tends to infinity?

[MA '23]

Let $s \in \mathbb{N}$ such that $s > 2 + \frac{1}{2}$, and $\kappa > 0$. Moreover, consider regular and controlled $(\underline{h}, \underline{u})$ on $(\rho_{surf}, \rho_{bott})$, and (h, u) a strong sufficiently regular solution to (**CSHS+GM**) on a time interval $[0, T]$ with $T > 0$, satisfying natural assumptions.

Then there exists $N_0(T, \kappa) \in \mathbb{N}^*$ such that for all $N \geqslant N_0$ and any initial data $(H_0, U_0) = (P_N(h_0), P_N(u_0)) \in H^s(\mathbb{R})^{2N}$ the solution to (**MSWS+GM**) with $\underline{H} = P_N(\underline{h})$, $\underline{U} = P_N(\underline{u})$ and satisfying $(H, U)_{t=0} = (H_0, U_0)$ defined in the previous Theorem is well-defined on the time interval $[0, T]$ and satisfies for any $t \in [0, T]$

$$|||(H - P_N h, U - P_N u)|||_s(t) = \mathscr{O}_{\kappa,T}\left(\frac{1}{N^2}\right),$$

where
$$\begin{aligned} P_N : \mathscr{C}[\rho_{surf}, \rho_{bott}] &\to \mathbb{R}^N \\ f &\mapsto P_N(f) = (f(\boldsymbol{\rho}_i))_{1 \leqslant i \leqslant N}. \end{aligned}$$

Tools of the proof:

- Consistency
  - A careful and precise analysis is done to obtain the rate $\frac{1}{N^2}$.
  - The same result can be obtained for other choices of the operator P_N for instance $(P_N(f))_i = \frac{1}{\rho_{i+1}-\rho_i} \int_{\rho_i}^{\rho_{i+1}} f(\rho) d\rho$ but we may loose will lose the following property $P_N(fg) = P_N(f)P_N(g)$.
- Stability estimates
  - When we consider the system of the difference between the solutions we obtain the same structure of equations as in the continuous case.
  - The estimates of this difference derive naturally from the estimates found in the continuous case [Duchêne & Bianchini '22].

Conclusion:

We rigorously justified the (**MSWS+GM**) as an approximation of the (**CSHS+GM**) when the number of layers N tends to infinity.

The "inverse" limit:

The convergence of (**CSHS+GM**) to (**MSWS+GM**) when we consider a continuous density that converges to a piecewise continuous density [Duchêne, Bianchini, Adim](2024)(ArXiv preprint).

Thank you for your attention