Splitting methods for the nonlinear Klein–Gordon equations in low regularity and conservation properties

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Outline

1 The Model

2 Discretization and main result

3 Scheme of the proof

We consider the nonlinear Klein-Gordon equations on the circle

$$\boxed{\partial_t^2 q - \partial_x^2 q + mq + g(q) = 0} \tag{KG}$$

where
$$\begin{cases} (x,t) \in \mathbb{T} \times \mathbb{R} \\ q = q(x,t) \in \mathbb{R} \\ \text{the mass } m > 0 \\ g \text{ is a smooth real non-linearity with } g(0) = g'(0) = 0. \end{cases}$$

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We assume that the initial data

$$(q_{|t=0}, \partial_t q_{|t=0}) = (q(0), \partial_t q(0))$$

is small in $H^s \times H^{s-1}$ for $s > 1/2$. In other words

$$\|q(0), \partial_t q(0)\|_{H^s \times H^{s-1}} := \varepsilon \ll 1.$$

We set $p := \partial_t q$ and introduce the operator $\Lambda := \sqrt{-\partial_x^2 + m}$ defined by

$$\Lambda q = \sum_{j \in \mathbb{Z}} q_j \Lambda e^{ijx} = \sum_{j \in \mathbb{Z}} q_j \omega_j e^{ijx}$$

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Equation (KG) is written as the Hamiltonian system

$$\partial_t \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\Lambda^2 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ -g(q) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla H(q,p),$$

$$\boxed{H(q,p) = \frac{1}{2} \int_{\mathbb{T}} \left(p^2 + (\Lambda^2 q) q\right) \mathrm{d}x + \int_{\mathbb{T}} G(q)\, \mathrm{d}x}$$

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- The Hamiltonian H is a constant of motion for this system

$$H(q(t), p(t)) = H(q(0), p(0)) \quad \forall t \in \mathbb{R}.$$

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- As a corollary, we expect to obtain a control of the dynamics of (KG)

$$\sum_{j\in\mathbb{Z}} \langle j\rangle^{2s}\mathcal{E}_j \sim_{s,m} \|\cdot\|^2_{H^s\times H^{s-1}}.$$

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Goal : To prove the almost preservation of $\mathcal{E}_j$ (at low regularity,) after applying the full discretizations considered by [Cohen–Hairer–Lubich08].

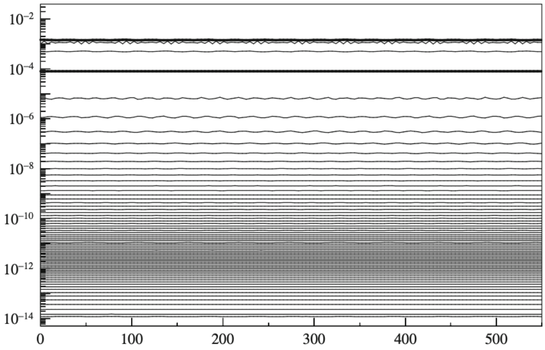
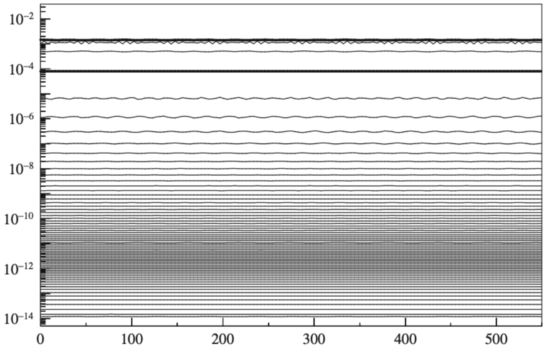


Figure: Numerical conservation properties for (KG). Adapted from *Conservation of energy, momentum and actions in numerical discretizations of non-linear wave equations*, 2008 by David Cohen, Ernst Hairer and Christian Lubich



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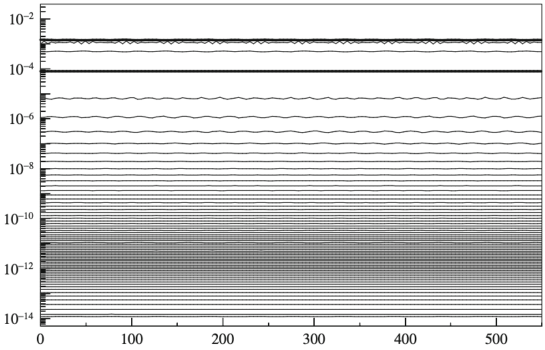

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- They considered initial data living in $H^3 \times H^2$.
- We prove this result for $s = 1$. i.e. for initial data in $H^1 \times L^2$.

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Consider a standard pseudo-spectral **semi-discretization** with equidistant collocation points

$$x_\ell = \frac{2\ell\pi}{K} \quad \text{where} \quad |\ell| < K/2 \quad \text{and } K \text{ is odd.}$$

This yields an approximation of the form

$$q(x,t) = \sum_{|j|<K/2} q_j e^{ijx} \quad \text{and} \quad p(x,t) = \sum_{|j|<K/2} p_j e^{ijx}.$$

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Taking the discrete Fourier transform $(\mathcal{F}^K\psi)_j = \frac{1}{K}\sum_{|\ell|<K/2} \psi_\ell e^{-ijx_\ell}$, the semi discretized (KG) equation can be written as a Hamiltonian system

$$\partial_t \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla H^K(q,p),$$

$$H^K(q,p) = \underbrace{\frac{1}{2}\sum_{|j|<K/2} \omega_j^2 |q_j|^2 + |p_j|^2}_{Z_2^K(q,p)} + \underbrace{\frac{1}{K}\sum_{|\ell|<K/2} G\left(q(x_\ell,t)\right)}_{P^K(q,p)}.$$

Fully discrete Splitting method

We split the equation into two systems

$$\partial_t \begin{pmatrix} q \\ p \end{pmatrix} = L \begin{pmatrix} q \\ p \end{pmatrix} \quad \text{and} \quad \partial_t \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ -g(q) \end{pmatrix}.$$

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Solving them, we get

$$\Phi^t_{Z_2^K}(q,p) = e^{tL} \begin{pmatrix} q \\ p \end{pmatrix} \quad \text{and} \quad \Phi^t_{P^K}(q,p) = \begin{pmatrix} q \\ p - tg(q) \end{pmatrix}.$$

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In our work, we consider the Strang splitting method

$$\Phi^h_{Z_2^K + P^K} \simeq \Phi^h_{\mathrm{num}} := \Phi^{h/2}_{P^K} \circ \Phi^h_{Z_2^K} \circ \Phi^{h/2}_{P^K}.$$

Remark. We can prove that it is one of the *symplectic mollified impulse* methods considered in [Cohen–Hairer–Lubich08].

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Let $\delta \in (0, 2\pi)$ and $r \geq 1$ arbitrarily large. For almost all $m > 0$, provided that $s \gtrsim r^2$ and under *CFL condition*

$$r\, h\, \omega_{(K-1)/2} \leq 2\pi - \delta,$$

there exists $\varepsilon_0 > 0$, if $\|q^0, p^0\|_{H^s \times H^{s-1}} := \varepsilon < \varepsilon_0$, we have

$$nh < \varepsilon^{-r} \implies \sum_{|j|<K/2} \langle j \rangle^{2s+1} \frac{|\mathcal{E}_j(q^n, p^n) - \mathcal{E}_j(q^0, p^0)|}{\varepsilon^2} \lesssim_{s,r,m} \varepsilon.$$

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Main flow: high regularity constraint $s \gtrsim r^2$

Theorem [Abou-Khalil–Bernier24]

Let $\delta \in (0, 2\pi)$ and $r \geq 1$ arbitrarily large. For almost all $m > 0$, provided that $s = 1$ and under *CFL condition*

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- We can only control the low harmonic actions ☹

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Thank you for your attention!

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- Due to local well-posedness in $H^s \times H^{s-1}$ for $s > 1/2$, the dynamics of (KG) remain close to the dynamics of the linearized equation for times of order ε^{-1}

$$|t| \ll \varepsilon^{-1} \implies \left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} - e^{Lt} \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \right\|_{H^s \times H^{s-1}} \ll \varepsilon.$$

Theorem (Bambusi03,Bambusi–Grébert06,Cohen–Hairer–Lubich08)

Let $r \geq 1$ arbitrarily large. For almost all $m > 0$, provided that $s \gtrsim r^2$, there exists $\varepsilon_0 > 0$, if $\|u(0), v(0)\|_{H^s \times H^{s-1}} := \varepsilon < \varepsilon_0$, the solution of (KG) satisfies for $|t| < \varepsilon^{-r}$

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- Similar stability results of this kind:
  quantum harmonic oscillators [Grébert–Imekraz–Paturel09],
  NLS on flat tori [Bambusi–Feola–Montalto22],
  capillary-gravity periodic water waves [Berti–Delort17] ...

The theorem states:

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Let $r \geq 1$ arbitrarily large. For almost all $m > 0$, provided that $s = 1$, there exists $\varepsilon_0 > 0$, if $\|u(0), v(0)\|_{H^1 \times L^2} := \varepsilon < \varepsilon_0$, the solution of (KG) satisfies for $|t| < \varepsilon^{-r}$

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- $\beta_r \gg 1$ is a constant depending only on r (very large).
- Initial data live in the energy space ($s = 1$).
- We cannot control all the super-actions.
- Similar results can be obtained for many Hamiltonians; For instance [Abou-Khalil22] on the quantum harmonic oscillators on $\mathbb{R}$

$$i\partial_t u = (-\partial_{xx} + x^2 + V)u \pm |u|^{2p} u.$$

We start with backward error analysis in the spirit of [Faou–Grébert11] and [Faou's book] :

Propostion

Let $r \geq 1$ be arbitrarily large and $\varepsilon_0 > 0$ be fixed. Provided that the CFL condition is satisfied, there exists a real modified Hamiltonian H_h^K such that for all $\varepsilon < \varepsilon_0$ and $\|u, v\|_{H^1 \times L^2} := \varepsilon < \varepsilon_0$, we have

$$\left\|(\phi^h_{\mathrm{num}} - \phi^h_{H_h^K})(u, v)\right\|_{H^1 \times L^2} \lesssim_{r, \varepsilon_0, \delta} h^2 \varepsilon^{r+1}$$

and

$$|H_h^K(u, v) - H^K(u, v)| \lesssim_{\delta, m} h\varepsilon^3.$$

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Propostion

Let $r \geq 1$ be arbitrarily large and $\varepsilon_0 > 0$ be fixed. Provided that the CFL condition is satisfied, there exists a real modified Hamiltonian H_h^K such that for all $\varepsilon < \varepsilon_0$ and $\|u, v\|_{H^1 \times L^2} := \varepsilon < \varepsilon_0$, we have

$$\left\| (\phi^h_{\mathrm{num}} - \phi^h_{H_h^K})(u, v) \right\|_{H^1 \times L^2} \lesssim_{r, \varepsilon_0, \delta} h^2 \varepsilon^{r+1}$$

and

$$|H_h^K(u, v) - H^K(u, v)| \lesssim_{\delta, m} h \varepsilon^3.$$

- The modified Hamiltonian is constructed and is of the form

$$H_h^K = Z_2^K + Z_{3,h}^K + \cdots + Z_{r,h}^K.$$

Then, we prove the almost preservation of the energy and deduce a control of the numerical flow in the energy space in the spirit of [Faou–Grébert11] and [Gauckler17]:

Corollary

Provided that $\|u^0, v^0\|_{H^1\times L^2} := \varepsilon < \varepsilon_0$, we have for $nh \leq \varepsilon^{-r}$

$$|H^K(u^n, v^n) - H^K(u^0, v^0)| \lesssim_{\delta,r,m,\varepsilon_0} h\,\varepsilon^3$$

and

$$\|u^n, v^n\|_{H^1\times L^2} \lesssim_m \varepsilon.$$

Finally, we apply the Birkhoff normal form procedure in low regularity in the spirit of [Bernier–Grébert21]:

Theorem

Let $r \geq 1$ be arbitrarily large. There exists $\beta_r \gg 1$, for almost all $m > 0$, provided that CFL is satisfied, we can find $\varepsilon_1 \gtrsim \langle j \rangle^{-\beta_r}$ and a symplectic transformation τ defined on $B(0, \varepsilon_1)$ such that

$$H_h^K \circ \tau = Q + R$$

where Q commutes with the low super-actions $\mathcal{E}_j$ and R satisfies

$$|R(u,v)| \lesssim_{r,\delta,m} \langle j \rangle^{\beta_r} \|u,v\|_{H^1 \times L^2}^{r+2}.$$

Table of Comparisons

	[CHL08]	[A-KB24]
$H^s \times H^{s-1}$	$s \gg 1$	Energy space $H^1 \times L^2$
Super-actions	Low and high	Low
NR condition	Standard	Strong
CFL	$r\,h\omega_{(K-1)/2} \leq 2\pi - \delta$	$r\,h\omega_{(K-1)/2} \leq 2\pi - \delta$
Mass m	For almost all $m > 0$	For almost all $m > 0$

On the strong non resonance condition:

Let $q \geq 3$, $j_1 > \cdots > j_q \geq 0$, $\ell_1, \cdots, \ell_q \in \mathbb{Z}^*$ and set

$$\Omega_{\ell,j} = \ell_1 \omega_{j_1} + \cdots + \ell_q \omega_{j_q}$$

Consider the following *non resonance conditions*

- Weak $|\Omega_{\ell,j}| \gtrsim_\ell \langle j_1 \rangle^{-\beta_q}$
- Classical $|\Omega_{\ell,j}| \gtrsim_\ell \langle j_3 \rangle^{-\beta_q}$ [Bambusi03]
- Strong $|\Omega_{\ell,j}| \gtrsim_\ell \langle j_q \rangle^{-\beta_q}$ [Bernier–Grébert21]

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Theorem (Bernier–Grébert21)

For almost all $m > 0$, the strong non resonance condition is satisfied

Proof : Weak NR + $\lim_{j \to 0} \mathrm{d}(\omega_j, \mathbb{Z}) = 0$.

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Thank you for your attention!