NUMERICAL METHODS
FOR THE
FREE-SURFACE NAVIER-STOKES EQUATIONS
AND APPLICATIONS TO
BREAKING WATER WAVES

CANUM 2024

MODÉLISATION, MÉTHODES NUMÉRIQUES ET APPLICATIONS EN OCÉANOGRAPHIE

Alan Riquier - Département de Mathématiques et Applications (ENS - PSL)

Joint work with Emmanuel Dormy (DMA - ENS PSL)

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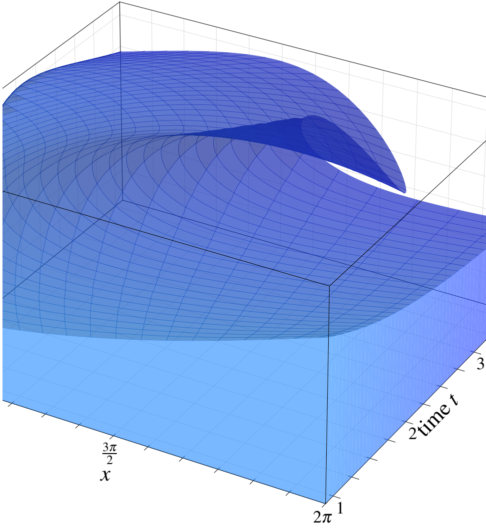
Outline

1. The free-surface Navier-Stokes equations
2. Finite element framework
3. The $\mathrm{Re} \to +\infty$ limit
4. The superficial boundary layer



Problem formulation

Navier-Stokes

Lagrangian advection

Initial condition

Viscous Water Waves

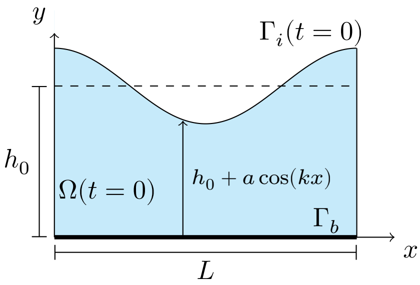
Nondimensionalization

Nondimensional quantities are defined as follows

$$\boldsymbol{x} \to h_0 \boldsymbol{x}$$
$$\boldsymbol{u} \to \sqrt{g h_0} \cdot \boldsymbol{u}$$
$$p \to \rho g h_0 \cdot p$$

This allows to define the **Reynolds number** Re,

$$\mathrm{Re} = \frac{\rho h_0 \sqrt{g h_0}}{\mu}$$



Viscous Water Waves

Navier-Stokes equation

Incompressible, non-dimensional, **Navier-Stokes** equation in $\Omega(t)$:

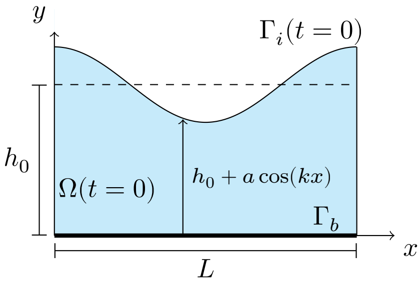
$$\left\{ \begin{array}{rcl} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} & = & -\nabla p + \frac{1}{\mathrm{Re}} \Delta \boldsymbol{u} + \boldsymbol{g} \\ \nabla \cdot \boldsymbol{u} & = & 0 \end{array} \right.$$

Navier boundary conditions on Γ_b,

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad ; \quad \boldsymbol{t} \cdot \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t\right] \cdot \boldsymbol{n} = 0$$

Stress-free boundary condition on $\Gamma_i(t)$,

$$p\boldsymbol{n} - \frac{1}{\mathrm{Re}} \cdot \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t\right] \cdot \boldsymbol{n} = 0$$



Weak formulation

Navier-Stokes problem

Function space

$$\mathbf{H}^1_{\Gamma_b}(\Omega(t)) = \left\{ \boldsymbol{v} \in H^1(\Omega(t); \mathbb{R}^2) : \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_b \right\}$$

We **do not** assume **the** incompressibility, $\nabla \cdot \boldsymbol{u} = 0$, directly in the function space as it would not work in finite element!

Find $\boldsymbol{u} \in \mathcal{C}^1([0,T); \mathbf{H}^1_{\Gamma_b})$ and $p \in L^\infty([0,T), L^2)$ such that

$$\boxed{\int_{\Omega(t)} \boldsymbol{v} \cdot \partial_t \boldsymbol{u} + \boldsymbol{v} \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \frac{2}{\mathrm{Re}} \mathbb{S}(\boldsymbol{v}) : \mathbb{S}(\boldsymbol{u}) - p \nabla \cdot \boldsymbol{v} + q \nabla \cdot \boldsymbol{u} - \boldsymbol{v} \cdot \boldsymbol{g} = 0}$$

for all $\boldsymbol{v} \in \mathbf{H}^1_{\Gamma_b}$ and $q \in L^2$, at all time $t \in (0,T)$.

Surface tension

Let Bo the **Bond number**,

$$\mathrm{Bo} = \frac{\rho g L^2}{\sigma}$$

and $\kappa : [0,T) \times \Gamma_s(t) \to \mathbb{R}$ the **surface curvature**.

Find $\boldsymbol{u} \in \mathcal{C}^1([0,T); \mathbf{H}^1_{\Gamma_b})$ and $p \in L^\infty([0,T), L^2)$ such that

$$\int_{\Omega(t)} \boldsymbol{v} \cdot \partial_t \boldsymbol{u} + \boldsymbol{v} \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \frac{2}{\mathrm{Re}} \mathbb{S}(\boldsymbol{v}) : \mathbb{S}(\boldsymbol{u}) - p \nabla \cdot \boldsymbol{v} + q \nabla \cdot \boldsymbol{u} - \boldsymbol{v} \cdot \boldsymbol{g} = \int_{\Gamma_s(t)} \kappa \mathrm{Bo}^{-1} \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}S$$

for all $\boldsymbol{v} \in \mathbf{H}^1_{\Gamma_b}$ and $q \in L^2$, at all time $t \in (0,T)$.

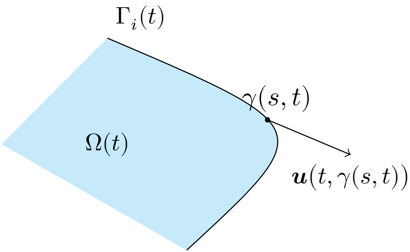
Interface advection

Lagrangian scheme

Interface is a **parametrised curve** $\gamma(s,t) \in \mathbb{R}^2$ whose evolution is given by

$$\frac{\partial \gamma}{\partial t}(s,t) = \boldsymbol{u}\Big(t, \gamma(s,t)\Big)$$

i.e. points on the interface have the same velocity as the fluid particles.



The interface contains all points parametrised by γ,

$$\Gamma_i(t) = \bigcup_s \{\gamma(s,t)\}$$

Initial conditions

Theory of linear waves

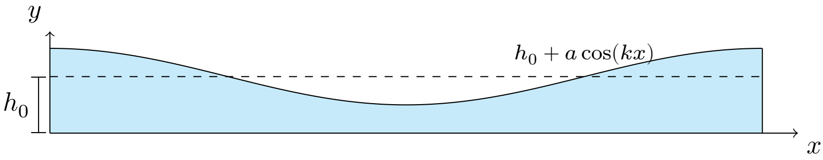
Linear wave of (small) amplitude a,

$$\gamma_0(t,x) = h_0 + a\cos(kx - \omega t) \qquad \textbf{with} \qquad \omega = \sqrt{gk\tanh(kh_0)}$$

$$\phi_0(t,x,y) = \frac{a\omega}{k}\frac{\cosh(ky)}{\sinh(kh_0)} \cdot \sin(kx - \omega t) + \mathcal{O}(ka)$$

The velocity is then

$$\boldsymbol{u}_0(t,x,y) = \nabla\phi_0 = \frac{a\omega}{\sinh(kh_0)} \cdot \begin{bmatrix} \cosh(ky)\cos(kx-\omega t) \\ \sinh(ky)\sin(kx-\omega t) \end{bmatrix}$$



Initial conditions

Laplace problem

$$\boldsymbol{u}_0(t,x,y) = \frac{a\omega}{\sinh(kh_0)} \cdot \begin{bmatrix} \cosh(ky)\cos(kx-\omega t) \\ \sinh(ky)\sin(kx-\omega t) \end{bmatrix}$$

so the velocity $\boldsymbol{n} \cdot \boldsymbol{u}_0$ in the **normal** direction can be computed from

$$\boldsymbol{n}_0(x) = \frac{1}{\sqrt{1+(\partial_x \gamma_0)}} \begin{bmatrix} -\partial_x \gamma_0 \\ 1 \end{bmatrix} \qquad \text{where} \quad \gamma_0(x) = h_0 + a\cos(kx)$$

The **initial velocity** is then constructed from assuming $y = h_0$ in $\boldsymbol{u}_0$ and solving the following Laplace problem

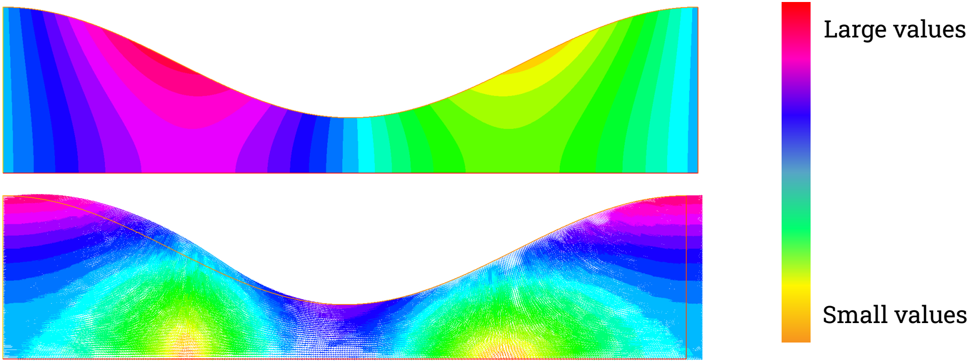
$$\begin{cases} \Delta\phi = 0 & \text{in the domain } \Omega(t) \\ \partial_n \phi = 0 & \text{at the bottom } \Gamma_b \\ \partial_n \phi = \boldsymbol{n}_0 \cdot \boldsymbol{u}_0(y = h_0) & \text{at the surface } \Gamma_i(t=0) \end{cases}$$

and the **initial velocity** is $\boldsymbol{u}(t=0) = \nabla\phi$.

Initial condition

Prescribed normal velocity

$$\partial_n \phi = \boldsymbol{n}_0 \cdot \boldsymbol{u}_0(y = h_0)$$



Numerical framework

Finite elements

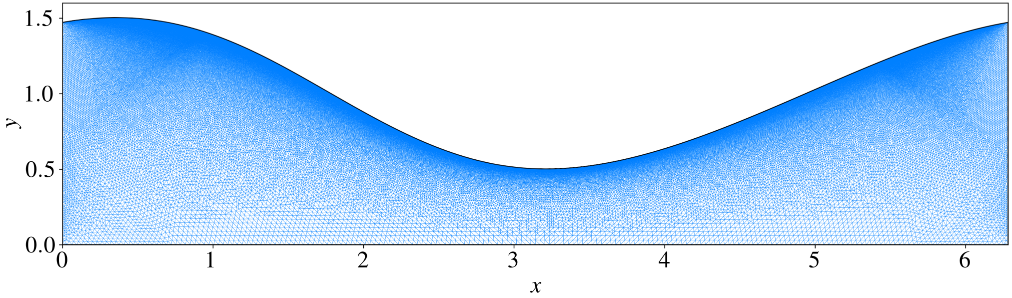
Mixed Lagrangian-Eulerian

Geometric multigrid solver

Finite Elements discretization

We use the `FreeFem` finite elements library Hecht (2012) for

- Mesh generation and handling
- Matrices computations and handling
- Interface with PETSc



$4\,000$ points on the interface, initially $\approx 200\,000$ triangles, $\approx 10^6$ degrees of freedom.

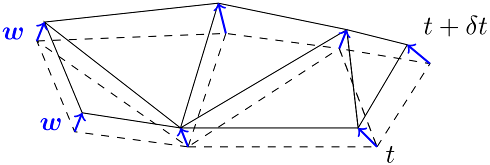
Mesh advection scheme

Let $\boldsymbol{w}$ the **velocity of the mesh**. At each time step, we numerically solve the problem

$$\begin{cases} \Delta \boldsymbol{w} = 0 & \text{in } \Omega_t \\ \boldsymbol{w} = \boldsymbol{u} & \text{on } \Gamma_{s,t} \\ \boldsymbol{w} = 0 & \text{on } \Gamma_b \end{cases}$$

And each point of the mesh is advected with velocity $\boldsymbol{w}$. Points on the interface are thus **purely Lagrangian**!

This is called the **Arbitrary Lagrangian Eulerian** method (ALE).



Time stepping scheme

Crank-Nicolson second order in time scheme. CFL condition to compute the time step at each iteration.

At each time step, we solve the following problem

$$\int_{\Omega(t)} \boldsymbol{v} \cdot \frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\delta t} + \boldsymbol{v} \cdot \left((\boldsymbol{u}^n - \boldsymbol{w}^n) \cdot \nabla \right) \left(\frac{\boldsymbol{u}^{n+1} + \boldsymbol{u}^n}{2} \right) + \frac{2}{\mathrm{Re}} \mathbb{S}(\boldsymbol{v}) : \mathbb{S} \left(\frac{\boldsymbol{u}^{n+1} + \boldsymbol{u}^n}{2} \right)$$
$$+ \left(\frac{p^{n+1} + p^n}{2} \right) \nabla \cdot \boldsymbol{v} + q \nabla \cdot \left(\frac{\boldsymbol{u}^{n+1} + \boldsymbol{u}^n}{2} \right) - \boldsymbol{v} \cdot \boldsymbol{g}$$
$$= 0$$

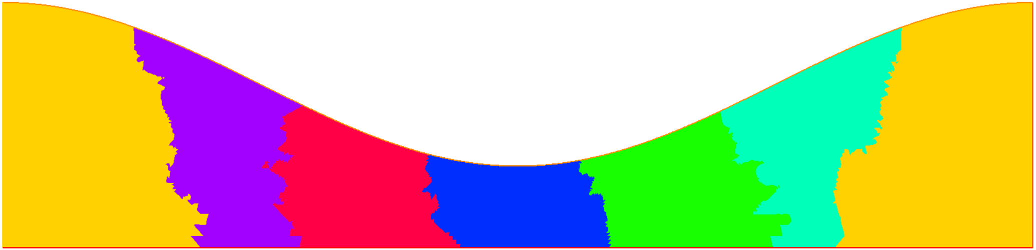
for all $(\boldsymbol{v}, q)$ and then compute $\boldsymbol{w}^{n+1}$ before advecting the mesh.

We use $\mathbb{P}^2$ elements for the velocity and $\mathbb{P}^1$ elements for the pressure.

Domain decomposition

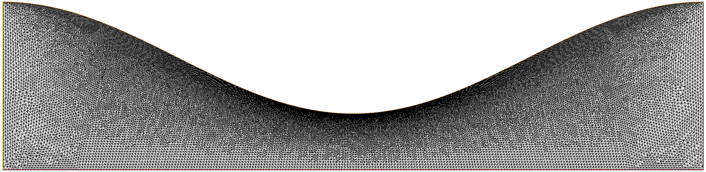
MPI domain decomposition with graph partitioner. **PETSc** matrices and solvers.

Example with **6 domains**:

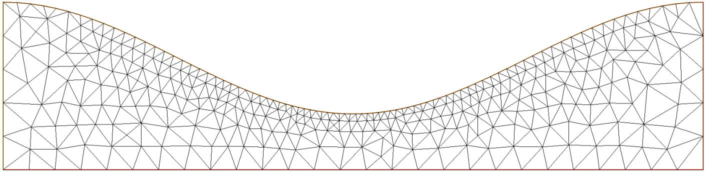


Geometric multigrid

We use a **geometric multigrid** solver for fast convergence using a large number of MPI processes.



Level 0 (fine)



Level 2 (coarse)

Timings and #dofs

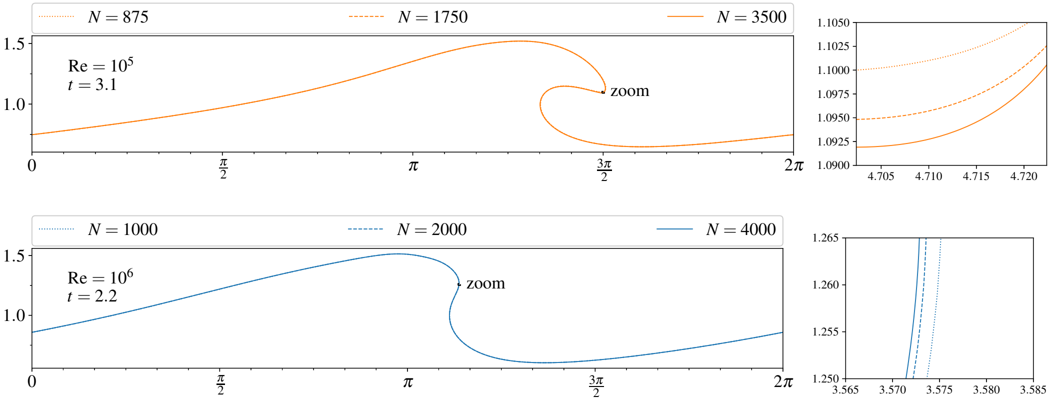
Between 1 and 3.5 million unknowns, convergence in $\sim 5 \pm 2$ GMRES iterations to machine precision $\rightarrow$ ~ 20 seconds on 48 CPU cores.

Main computational limitations due to **FreeFem memory management** and **spurious behaviors in mesh handling**.

Clearly this **method is not efficient enough to handle 3d**!

		# dofs at the **start**			# dofs at the **end**		
Re	N	# triangles	$(\boldsymbol{u}, p)$	$\boldsymbol{w}$	# triangles	$(\boldsymbol{u}, p)$	$\boldsymbol{w}$
10^2	3000	195,314	886,913	198,514	780,572	3,520,574	783,722
10^3	3500	221,748	1,007,116	225,448	615,382	2,675,923	596,294
10^4	3500	221,368	1,005,406	225,068	510,266	2,305,447	513,966
10^5	3500	222,970	1,012,615	226,670	450,648	2,037,166	454,348
10^6	4000	272,948	1,238,766	277,148	498,250	2,252,625	502,450

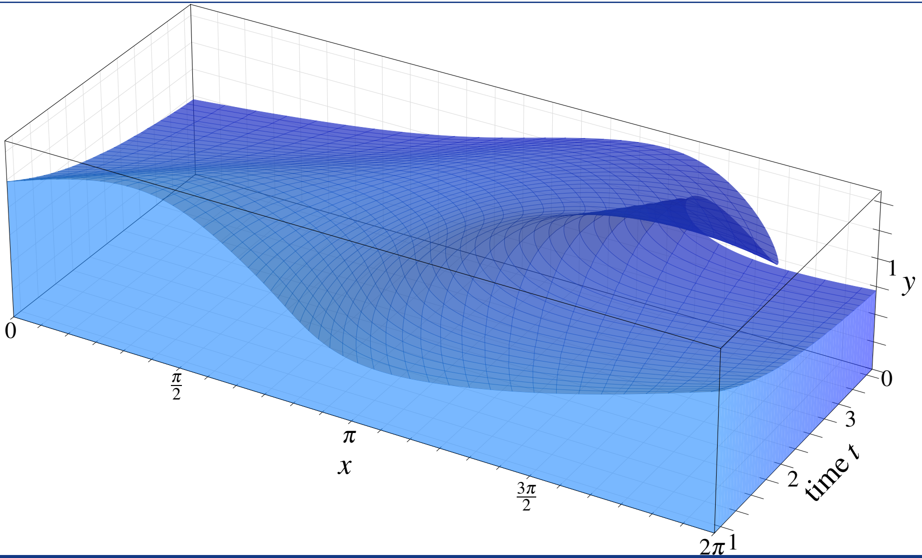
Convergence



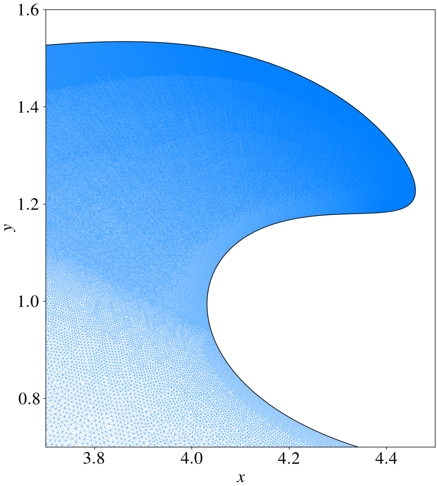
The $\mathrm{Re} \to +\infty$ limit

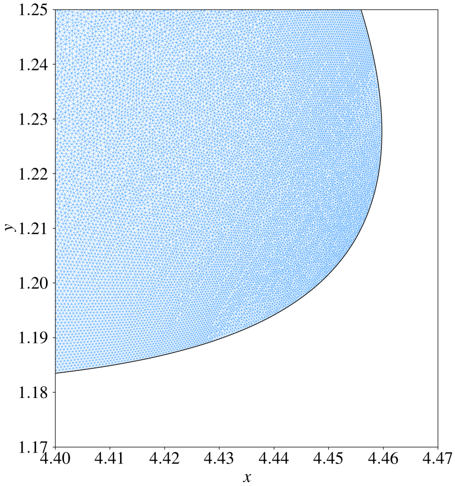
Comparing our results with the inviscid solution

$\mathrm{Re} = 10^6$ **result**

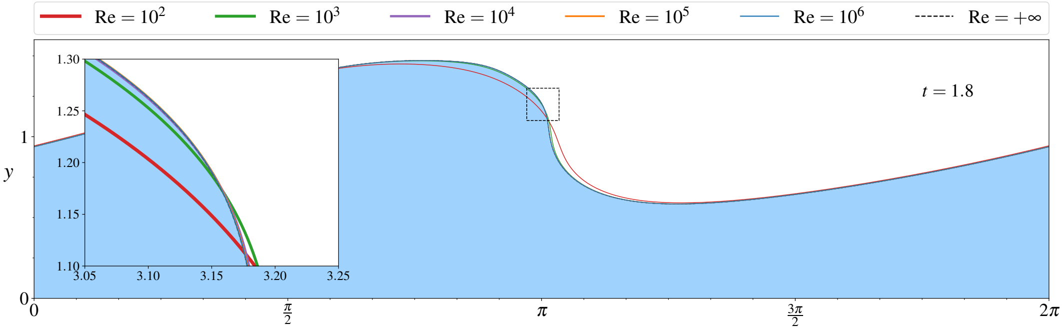


Mesh at $\mathrm{Re} = 10^6$



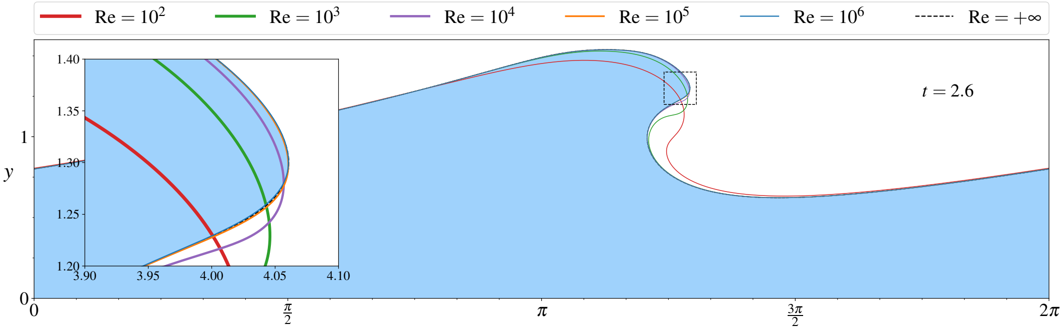


Interface for different values of Re



$\mathrm{Re} = +\infty$ simulations (i.e. Euler solution) computed with the numerical methods of Dormy & Lacave (2024).

Interface for different values of Re



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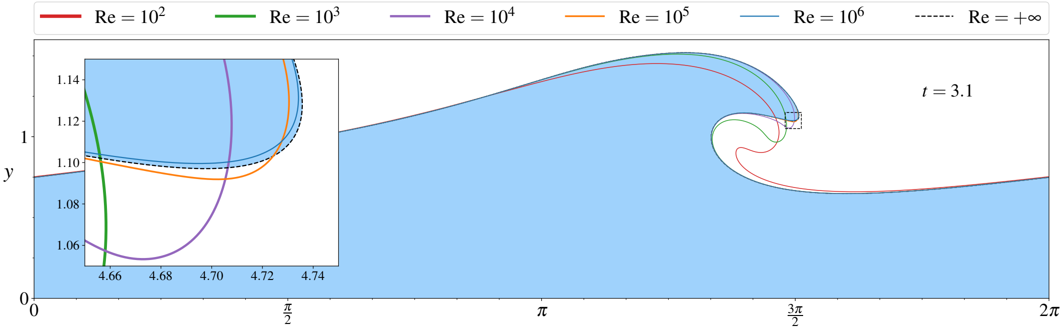

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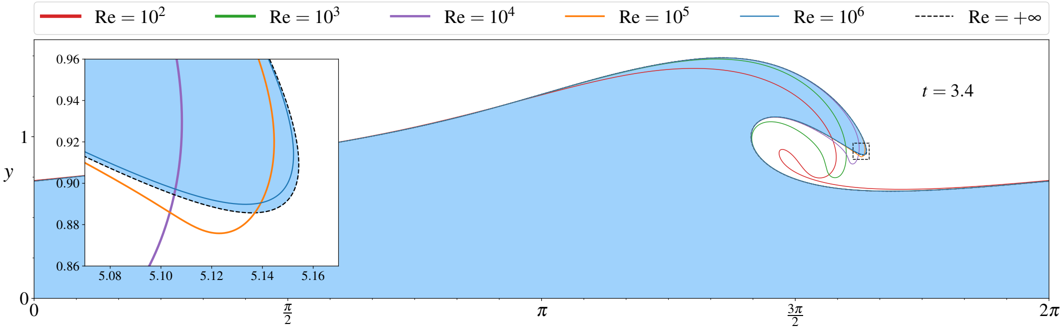

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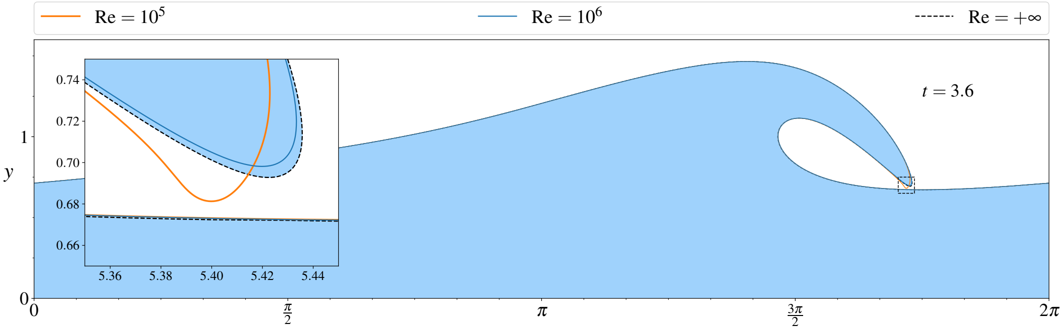
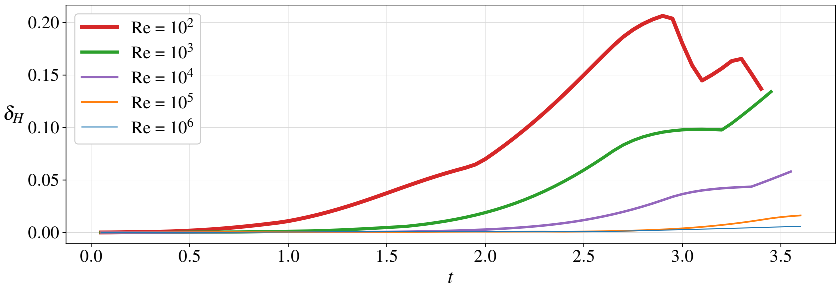


$\mathrm{Re} = +\infty$ simulations (i.e. Euler solution) computed with the numerical methods of Dormy & Lacave (2024).

Convergence
Haussdorff distance

To compare the Euler solution with Navier-Stokes, we use the **Haussdorff distance**,

$$\delta_H(\gamma_1, \gamma_2) = \max\left\{\tilde{\delta}_H(\gamma_1,\gamma_2), \tilde{\delta}_H(\gamma_2,\gamma_1)\right\} \quad \text{where} \quad \tilde{\delta}_H(\gamma_1,\gamma_2) = \max_{s_1}\min_{s_2}\left|\gamma_1(s_1) - \gamma_2(s_2)\right|$$



Boundary Layer

Where does the viscous dissipation happen?

Energy considerations

Link with vorticity

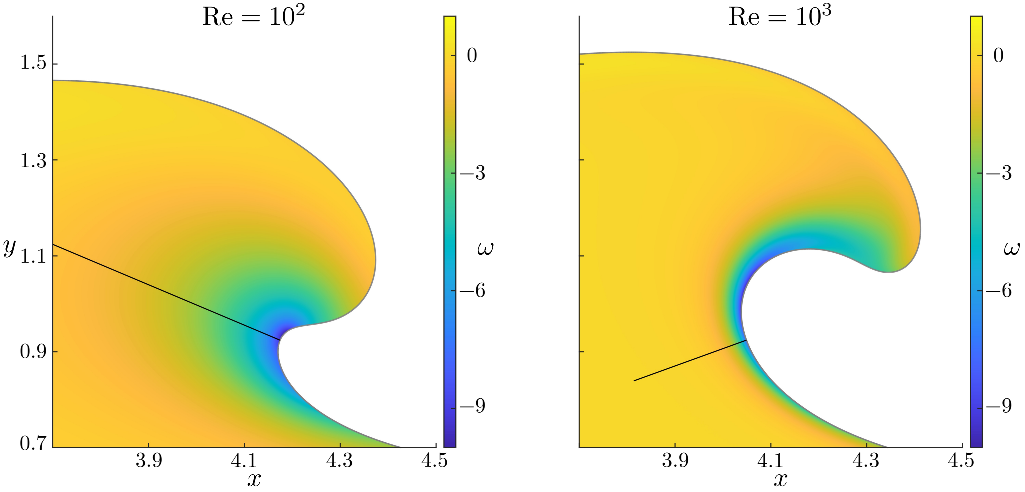
The **kinetic energy equation** can be obtained multiplying Navier-Stokes eq. by $\boldsymbol{u}$,

$$\partial_t \left(\frac{\boldsymbol{u}^2}{2} \right) = \boldsymbol{g} \cdot \boldsymbol{u} - \boldsymbol{u} \cdot \nabla p + \frac{1}{\mathrm{Re}} \left[\nabla \cdot (\boldsymbol{u}^\perp \omega) - \omega^2 \right]$$

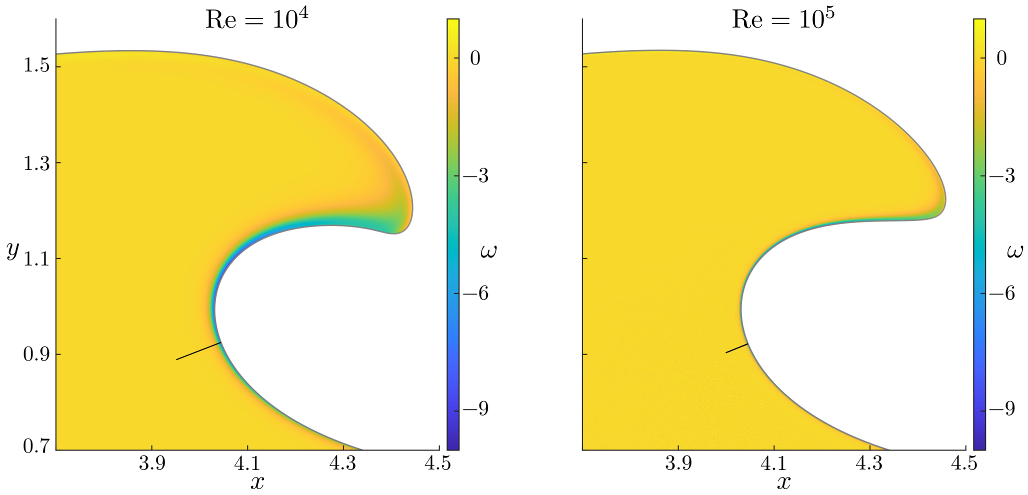
where $\boldsymbol{u}^\perp = [-u_y, u_x]$ and $\omega = \nabla^\perp \cdot \boldsymbol{u}$ is the **vorticity**.

This shows that fluids **dissipates energy** in **the support of** ω!

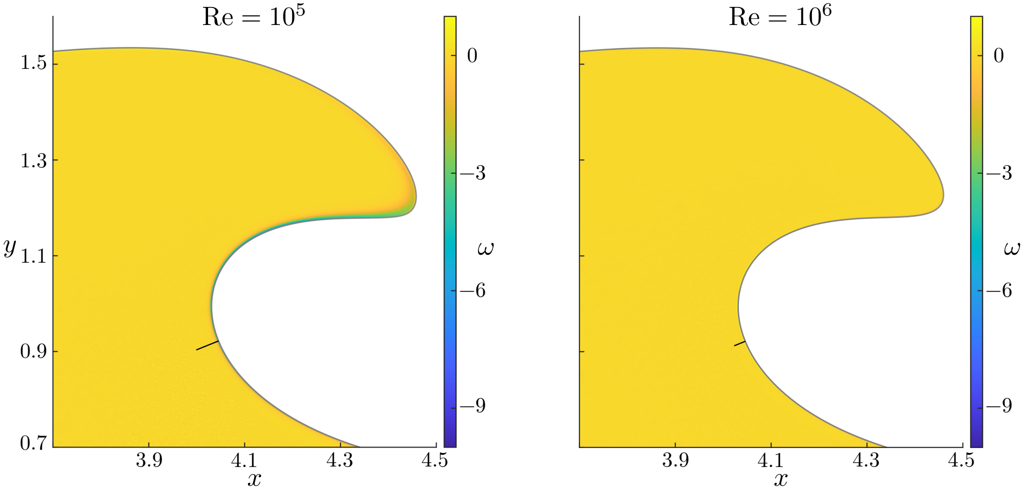
Viscous dissipation



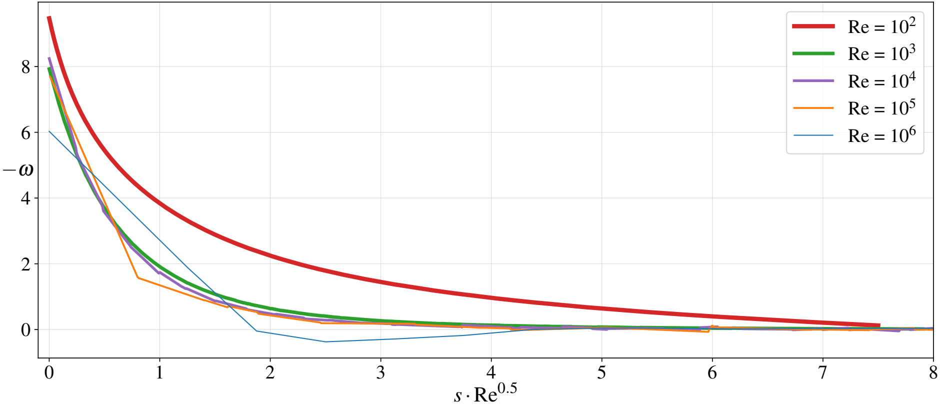
Viscous dissipation



Viscous dissipation



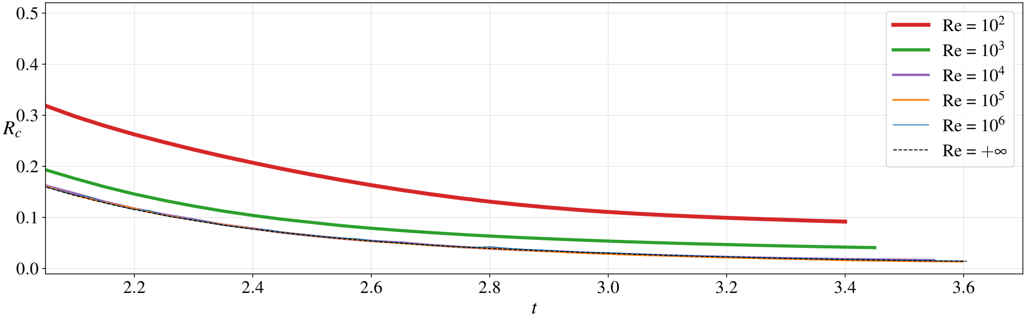
Size of the boundary layer



Exhibits a $\mathrm{Re}^{-\frac{1}{2}}$ scaling (usual in BL theory).

Thank you!

Maximum curvature of the interface



where $R_C = \kappa^{-1}$ is the curvature radius.