# A gluey contact model with friction for the numerical simulation of immersed granular media

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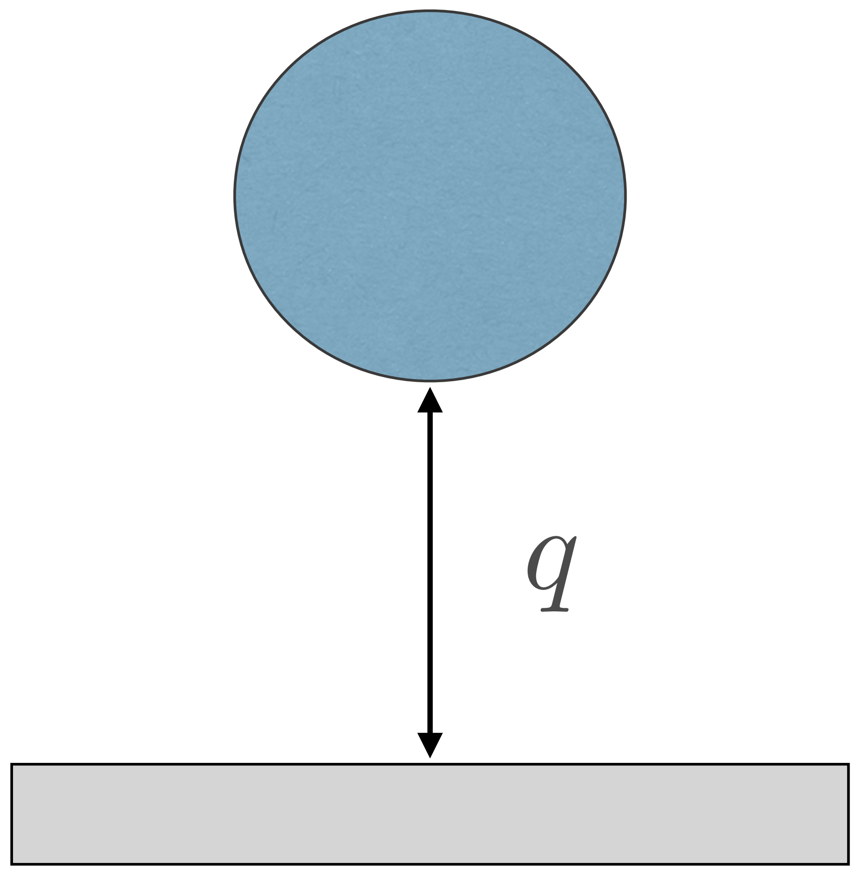
# A stiff problem.

Erosion

Lubrication force




$$F_{lub}(q) = -6\pi\mu r^2 \frac{\dot{q}}{q}$$

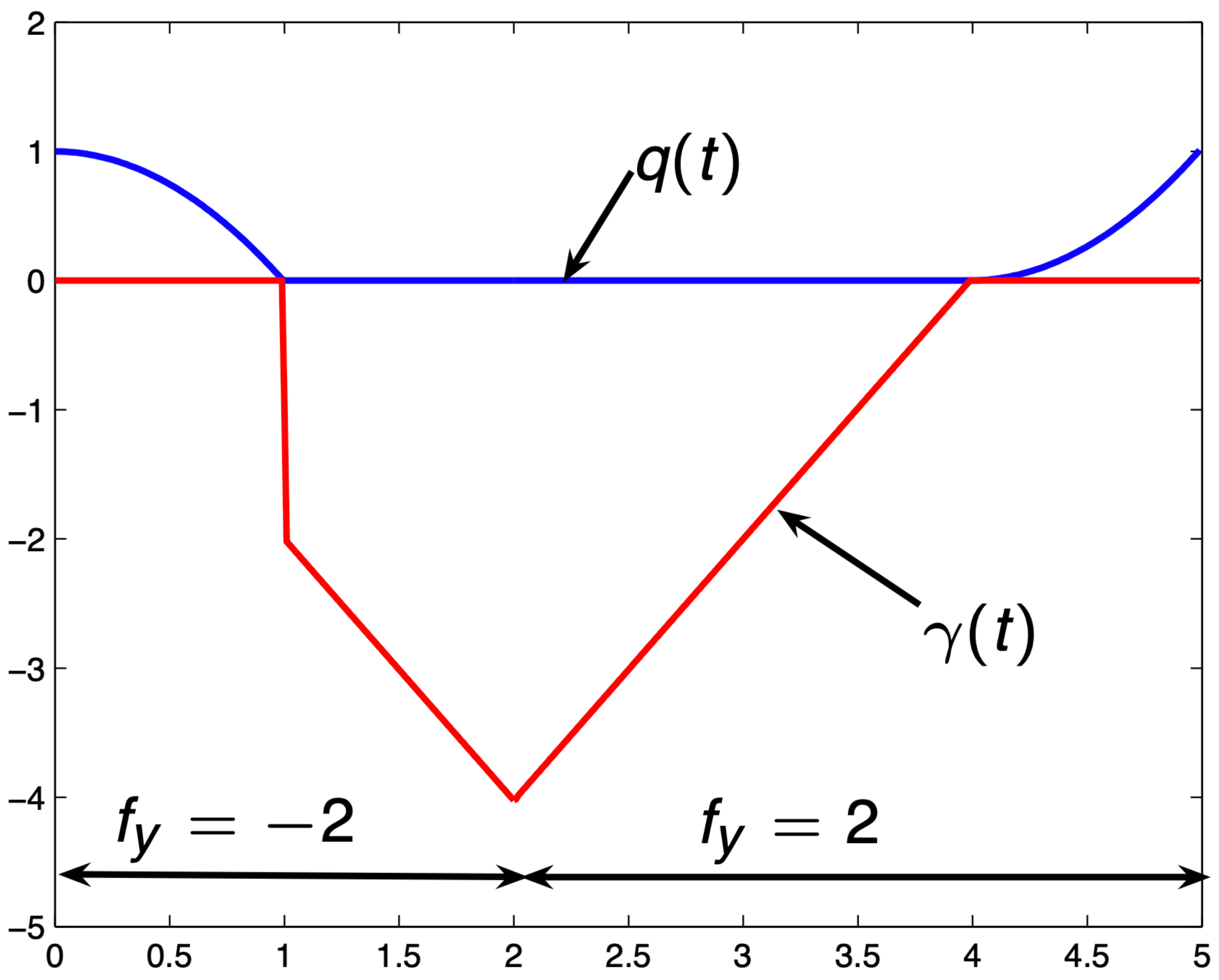
[Cox, 1974]

$$m\ddot{q} = -6\pi\mu r^2 \frac{\dot{q}}{q} + m f_y$$

- $q(t) > 0$ for any $t$
- BUT very small…

• Cox, R. G. (1974). The motion of suspended particles almost in contact. *International Journal of Multiphase Flow*, *1*(2), 343-371.

# Modeling lubrication: the gluey contact model.




$$\dot{q}^+ = P_{C_{q,\gamma}} \dot{q}^-$$

$$m\ddot{q} = m f_y + \lambda$$

$$\mathrm{supp}(\lambda) \subset \{t, q(t) = 0\}$$

$$\dot{\gamma} = -\lambda$$

$$q \geq 0, \gamma \leq 0$$

$$C_{q,\gamma} = \begin{cases} \{0\} & \text{si} \quad \gamma^- < 0 \\ \mathbb{R}^+ & \text{si} \quad \gamma^- = 0 \\ & \qquad\quad q = 0 \\ \mathbb{R} & \text{sinon} \end{cases}$$


- B. Maury, A gluey particle model, ESAIM Proceedings, July 2007, Vol.18, 133-142
- A. Lefebvre, Numerical simulation of gluey particles, M2AN, 43:53-80 (2009)

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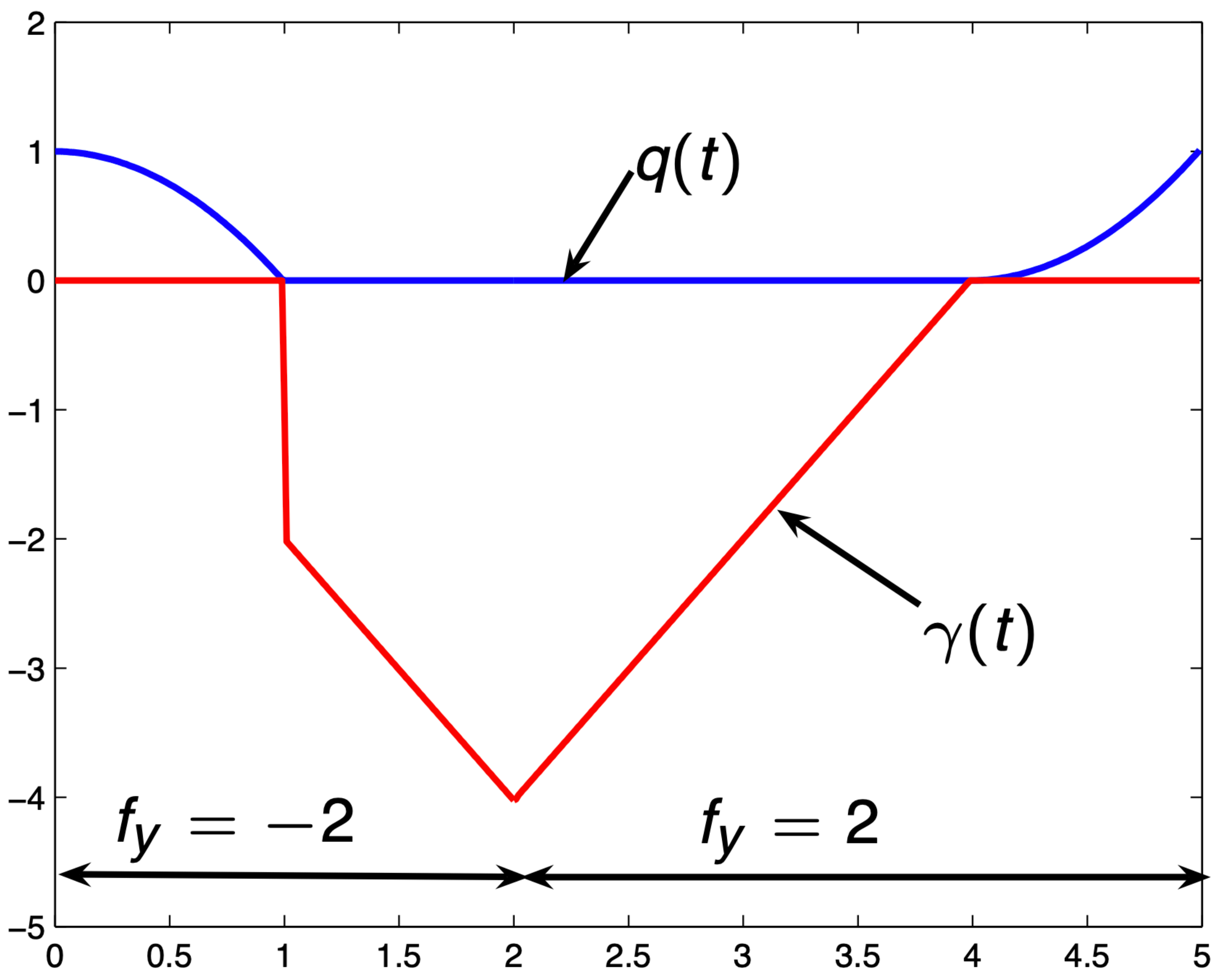


$$\dot q^+ = P_{C_{q,\gamma}} \dot q^-$$

$$m\ddot q = m f_y + \lambda$$

$$\mathrm{supp}(\lambda) \subset \{t, q(t) = 0\}$$

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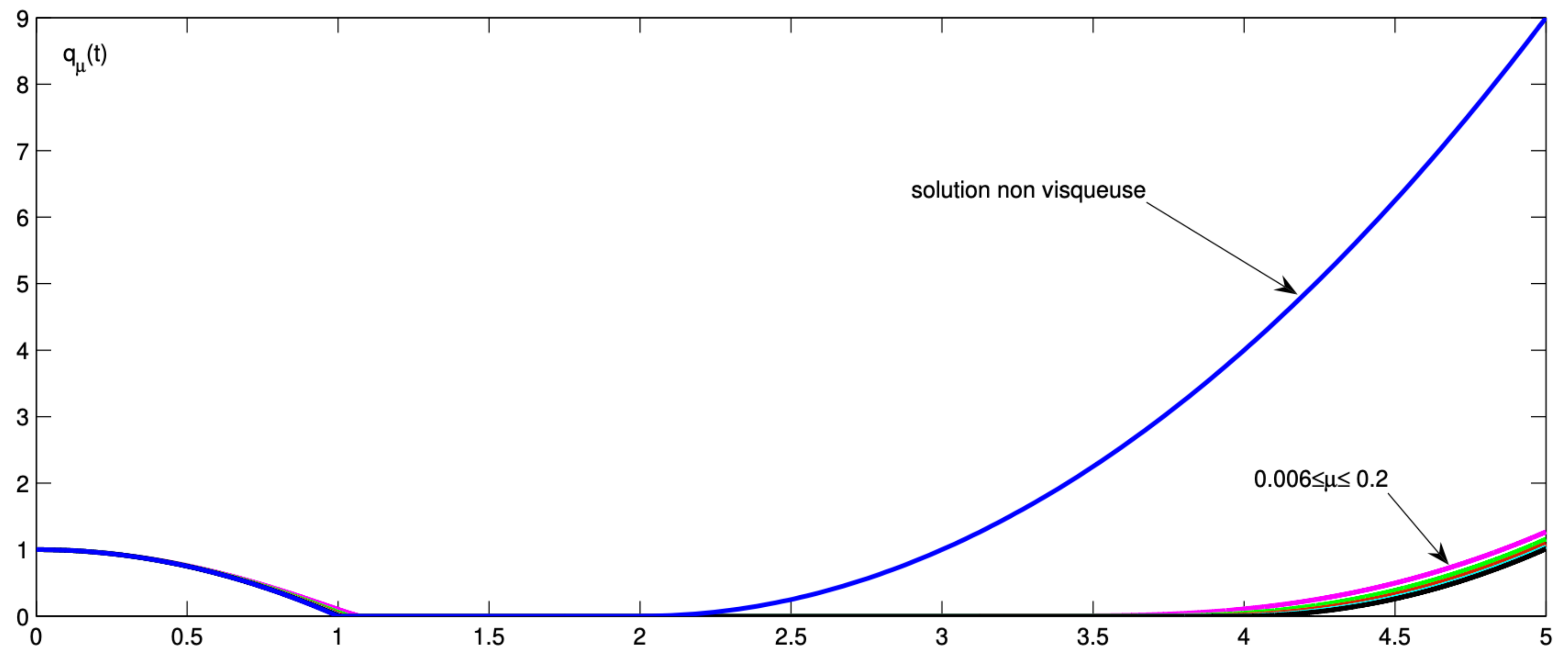
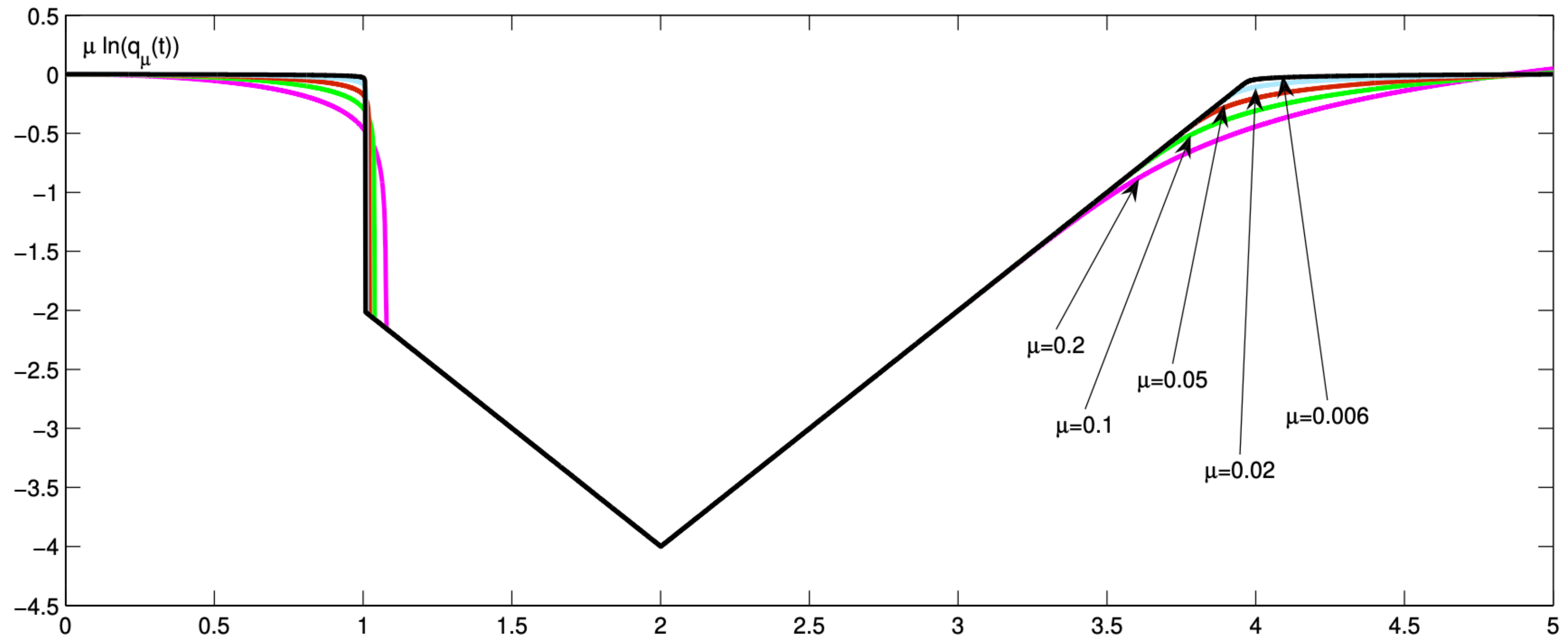
$$\mu \to 0$$



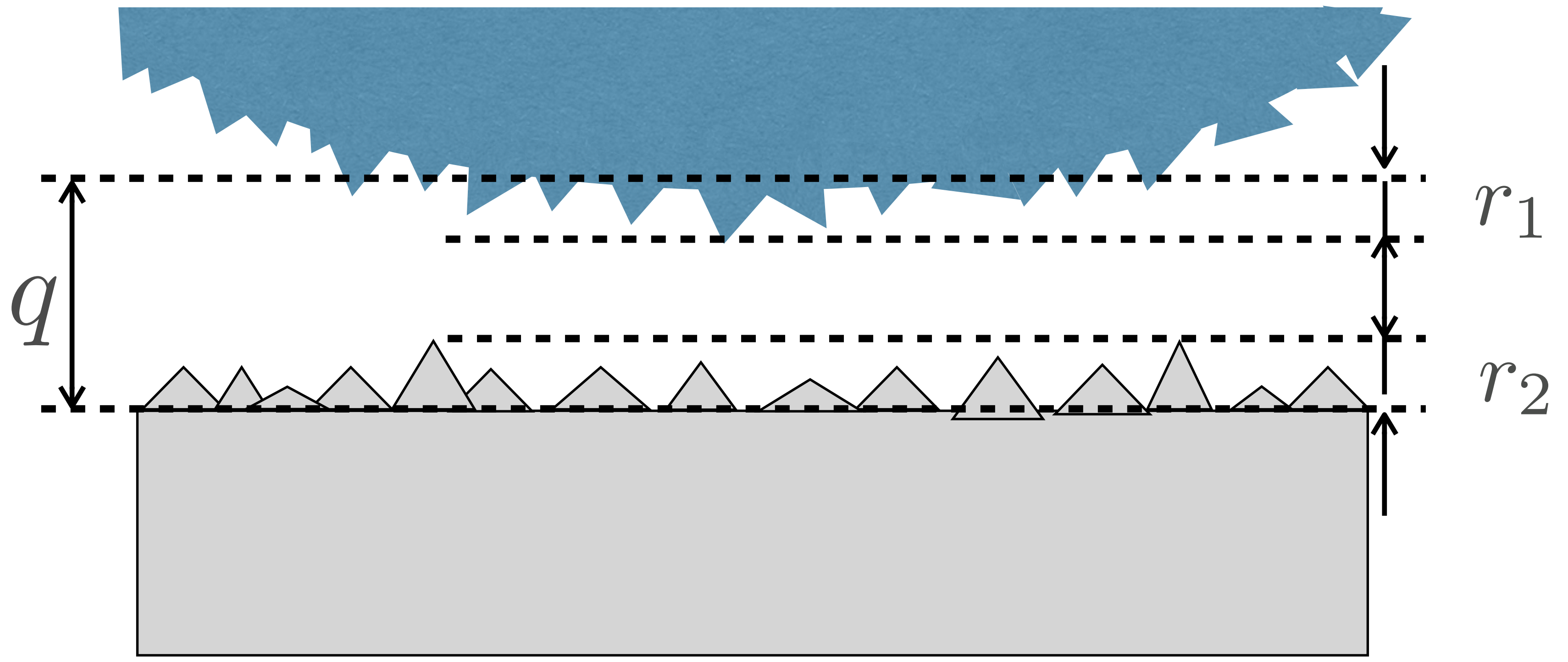




$$f_y = -2\,\mathbf{1}_{[0,2]} + 2\,\mathbf{1}_{[2,+\infty]}$$

$$\ddot q_\mu = -\mu \frac{\dot q_\mu}{q_\mu} + f_y$$

$$\gamma_\mu = \mu ln(q)$$

# Modeling lubrication: the gluey contact model.




- Roughness $\Longrightarrow$ contact

- Model : contact for $q = r_1 + r_2$

$\Longrightarrow$ Threshold for $\gamma$:

$$\gamma \geq \gamma_{min} = \mu \ln(r_1 + r_2)$$

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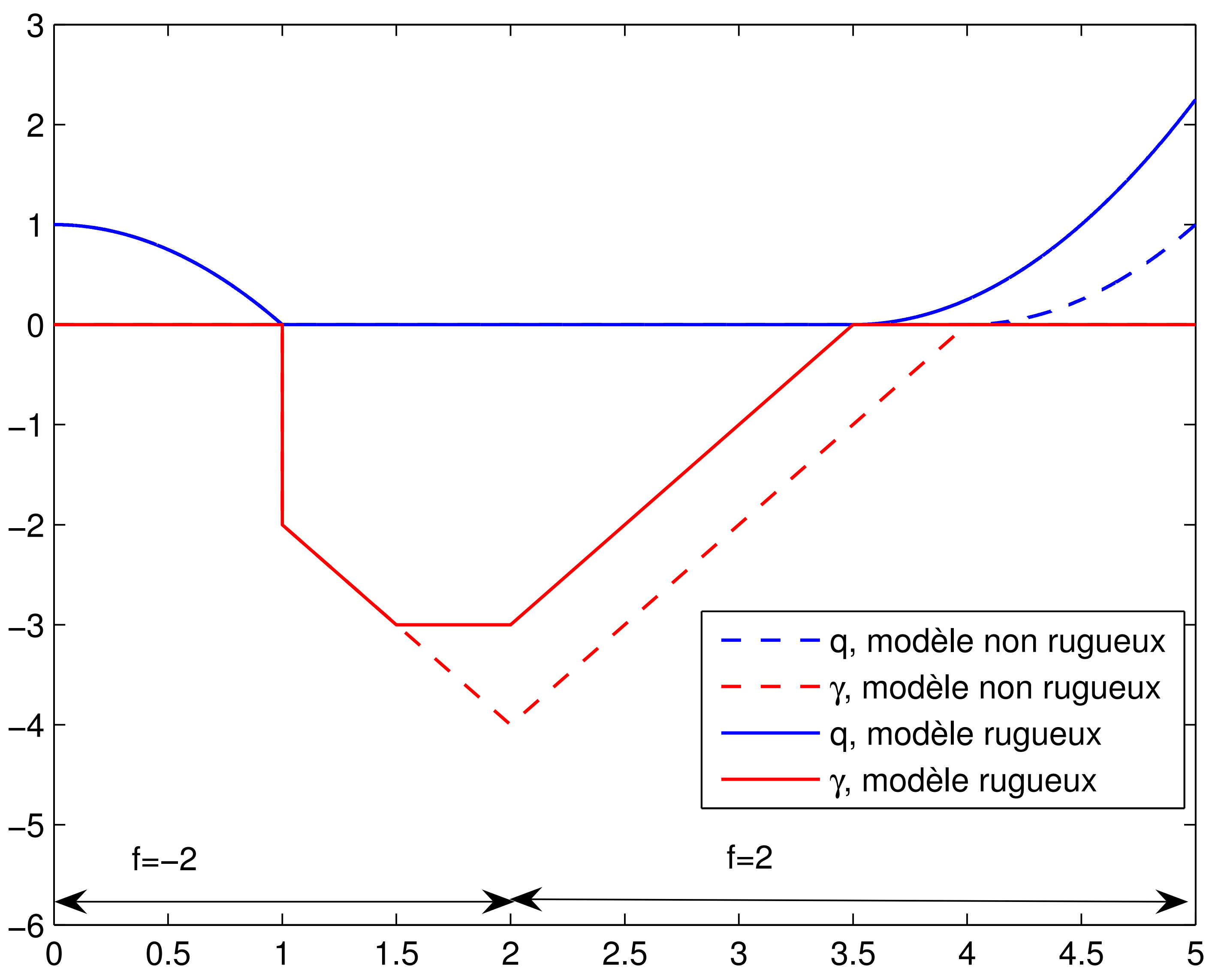


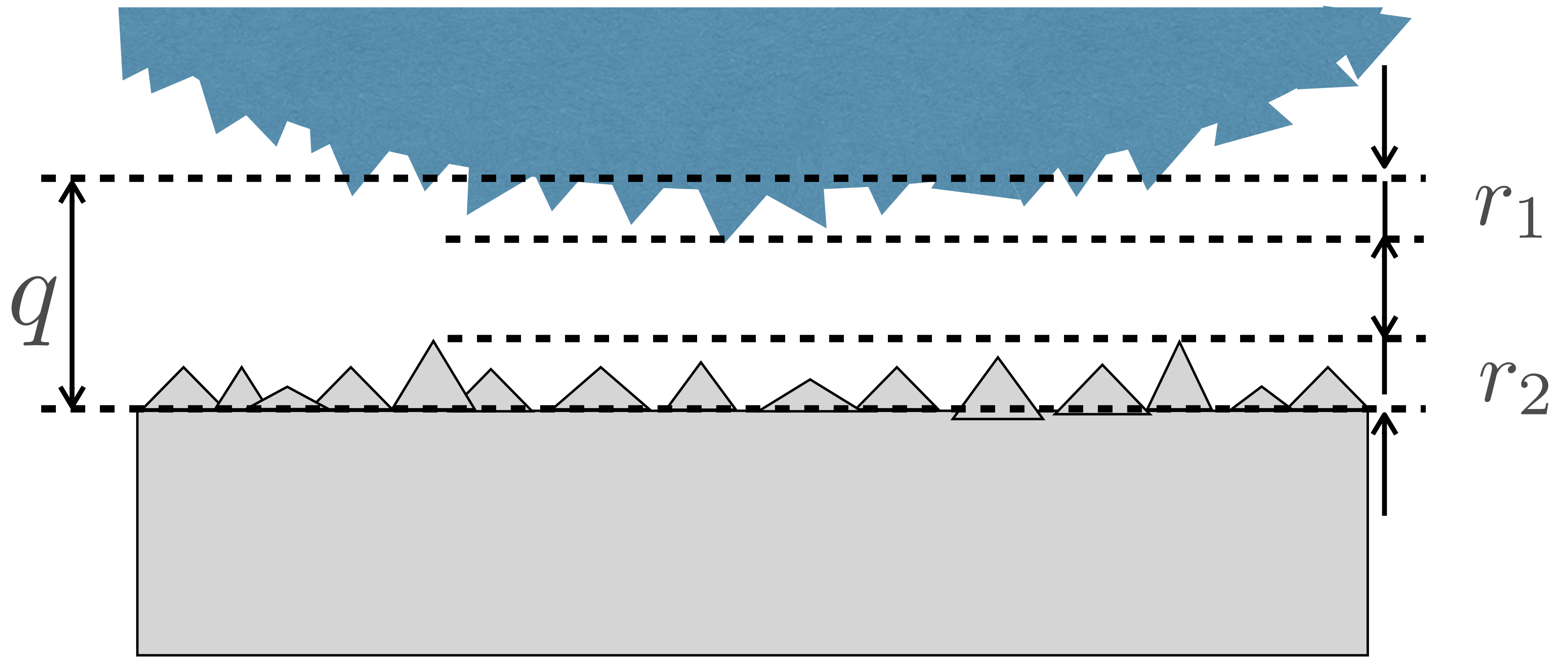


$$\dot{q}^+ = P_{C_{q,\gamma}} \dot{q}^-$$

$$m\ddot{q} = mf_y + \lambda$$

$$\mathrm{supp}(\lambda) \subset \{t, q(t) = 0\}$$

$$\mathrm{supp}(\delta) \subset \{t, \gamma(t^+) = \gamma_{min}\}$$

$$\dot{\gamma} = -\lambda + \delta$$

$$q \geq 0, \gamma_{min} \leq \gamma \leq 0, \delta \geq 0$$

$$C_{q,\gamma} = \begin{cases} \{0\} & \text{si} \quad \gamma^- < 0 \\ \mathbb{R}^+ & \text{si} \quad \gamma^- = 0 \\ & \qquad q = 0 \\ \mathbb{R} & \text{sinon} \end{cases}$$

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# Modeling lubrication: A numerical problem

A discrete minimisation problem $\dot{q}^+ = P_{C_{q,\gamma}} \dot{q}^-$

$$\min_{\mathbf{u} \in K^k} J(\mathbf{u})$$

$$J(\mathbf{u}) = \frac{1}{2} |\mathbf{u} - \mathbf{U}^{k+1}|^2, \ \mathbf{U}^{k+1} = \mathbf{u}^k + \Delta t \frac{1}{m} \mathbf{F}^k$$

$$K^k = \mathbf{u} \in \mathbb{R}^2 \text{ s t } \left| \begin{array}{l} -D - \Delta t \mathbf{u_n} \leq 0 \\ D + \Delta t \mathbf{u_n} \leq 0, \text{ si } \gamma^k < 0 \end{array} \right.$$

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$\Longrightarrow$

A dual problem

$$\min_{(\lambda^+, \lambda^-) \in K} J(\lambda^+, \lambda^-)$$

$$\mathbf{u}_\lambda = \mathbf{U}^{k+1} + \Delta t \tfrac{1}{m} (\lambda^+ - \lambda^-).\mathbf{n}$$

$$J(\lambda^+, \lambda^-) = (\lambda^+ - \lambda^-)[(D^k + \Delta t (\tfrac{\mathbf{u}_{\lambda n} + \mathbf{U_n}}{2})] + cst$$

$$K = \{(\lambda^+, \lambda^-), \ \lambda^+ \geq 0, \ \lambda^- \geq 0\}$$

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# Modeling Friction : An analogous problem

The dual problem for friction

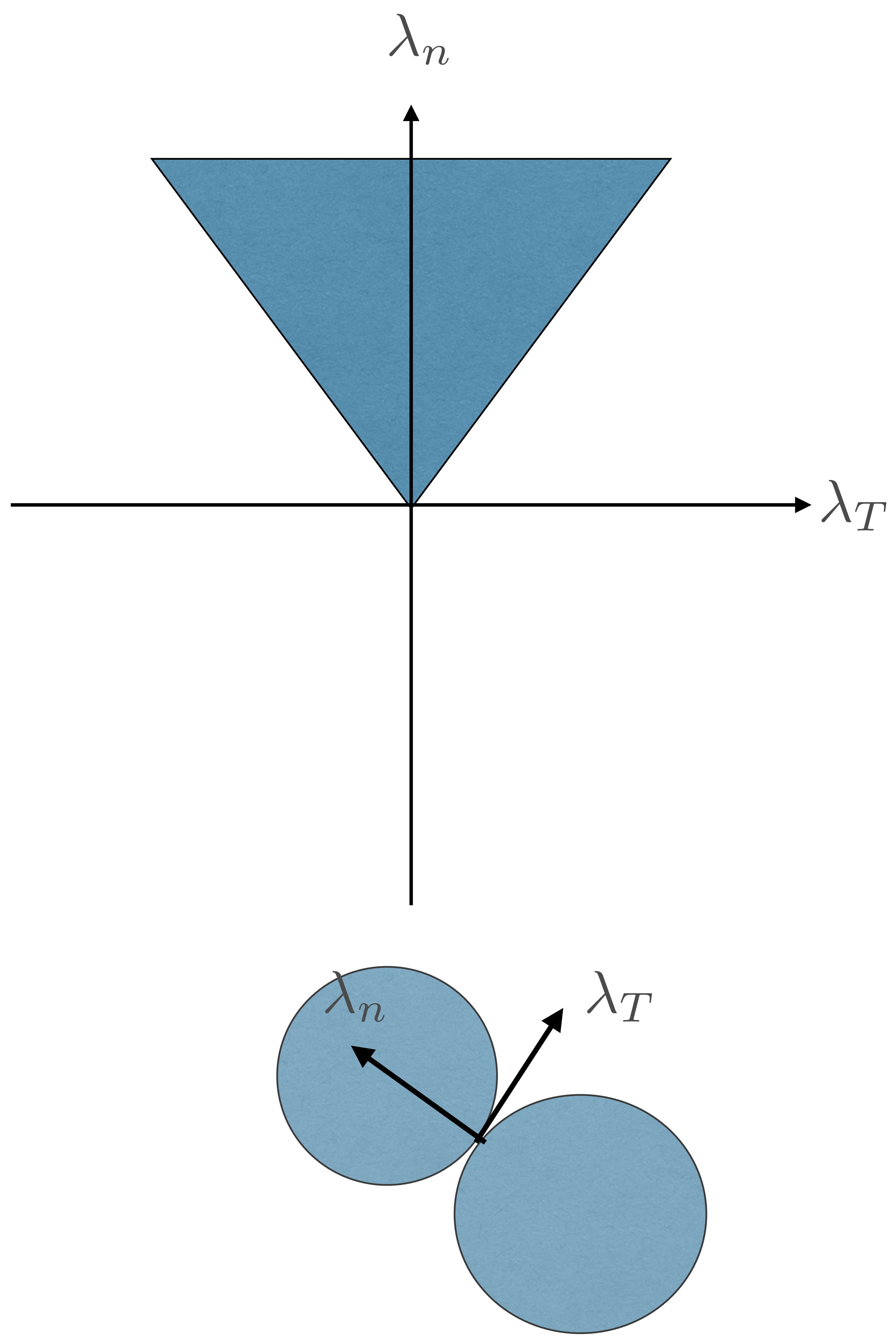
$$\min_{(\lambda_n,\lambda_T)\in K^k} J(\lambda_n,\lambda_T)$$

$$\mathbf{u}_\lambda = \mathbf{U}^{k+1} + \Delta t \frac{1}{m}(\lambda_n.\mathbf{n} + \lambda_\mathbf{T})$$

$$J(\lambda_n,\lambda_T) = \lambda_n[(D^k + \Delta t(\frac{\mathbf{u_{\lambda n}} + \mathbf{U}_n}{2})] + \langle \lambda_T, \Delta t \frac{\mathbf{u_{\lambda T}} + \mathbf{U}_T}{2})\rangle$$

$$K^k = \{(\lambda_n,\lambda_T),\ \mu\lambda_n \geq |\lambda_T|\}$$

The convex constraint




- D.E. Stewart. Convergence of a Time-Stepping Scheme for Rigid-Body Dynamics and Resolution of Painleve's Problem. Archive for Rational Mechanics and Analysis
- M. Anitescu. Optimization-based simulation of nonsmooth rigid multibody dynamics. Mathematical Programming, 2006.

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The dual problem for friction

$$\underset{(\lambda_n,\lambda_T)\in K^k}{min} J(\lambda_n,\lambda_T)$$

$$\mathbf{u}_\lambda = \mathbf{U}^{k+1} + \Delta t \tfrac{1}{m}(\lambda_n.\mathbf{n} + \lambda_\mathbf{T})$$

$$J(\lambda_n,\lambda_T) = \lambda_n[(D^k + \Delta t(\tfrac{\mathbf{u_{\lambda n}}+\mathbf{U}_n}{2})] + \langle \lambda_T, \Delta t \tfrac{\mathbf{u_{\lambda T}}+\mathbf{U}_T}{2})\rangle$$

$$K^k = \{(\lambda_n,\lambda_T),\ \mu\lambda_n \geq |\lambda_T|\}$$

$\Longrightarrow$

The primal problem

$$\underset{\mathbf{u}\in K^k}{min} J(\mathbf{u})$$

$$J(\mathbf{u}) = \frac{1}{2}|\mathbf{u}-\mathbf{U}^{k+1}|^2,\ \mathbf{U}^{k+1} = \mathbf{u}^k + \Delta t\frac{1}{m}\mathbf{F}^k$$

$$K^k \quad = \{\mathbf{u}\in\mathbb{R}^2,\ D + \Delta t\mathbf{u_n} \geq \mu\Delta t|\mathbf{u_t}|\}$$


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# Modeling Friction : An analogous problem

The dual problem for friction with fixed point

$$\min_{(\lambda_n,\lambda_T)\in K^k} J(\lambda_n,\lambda_T)$$

$$\mathbf{u}_\lambda = \mathbf{U}^{k+1} + \Delta t \frac{1}{m}(\lambda_n.\mathbf{n} + \lambda_\mathbf{T})$$

$$J(\lambda_n,\lambda_T) = \lambda_n[(D^k + s + \Delta t(\frac{\mathbf{u_{\lambda n}} + \mathbf{U}_n}{2})] + \langle \lambda_T, \Delta t \frac{\mathbf{u_{\lambda T}} + \mathbf{U}_T}{2})\rangle$$

$$K^k = \{(\lambda_n,\lambda_T),\ \mu\lambda_n \geq |\lambda_T|\}$$

$$F(s) = \mu\Delta t|\mathbf{u_T^s}|$$

$\Longrightarrow$

The primal problem with fixed point

$$\min_{\mathbf{u}\in K_s^k} J(\mathbf{u})$$

$$J(\mathbf{u}) = \frac{1}{2}|\mathbf{u} - \mathbf{U}^{k+1}|^2,\ \mathbf{U}^{k+1} = \mathbf{u}^k + \Delta t \frac{1}{m}\mathbf{F}^k$$

$$K_s^k = \{\mathbf{u} \in \mathbb{R}^2,\ D + \Delta t \mathbf{u_n} \geq \mu\Delta t|\mathbf{u_t}| - s\}$$

$$F(s) = \mu\Delta t|\mathbf{u_T^s}|$$


- D.E. Stewart. Convergence of a Time-Stepping Scheme for Rigid-Body Dynamics and Resolution of Painleve's Problem. Archive for Rational Mechanics and Analysis
- V. Acary, F. Cadoux, C. Lemarechal, and J. Malick. A formulation of the linear discrete coulomb friction problem via convex optimization. ZAMM-Journal of Applied Mathematics and Mechanics

# Modeling Viscosity and Friction : A non convex constraint

The dual problem for viscosity and friction with fixed point

$$\min_{(\lambda^-,\lambda_n,\lambda_T)\in K^k} J(\lambda^-,\lambda_n,\lambda_T)$$

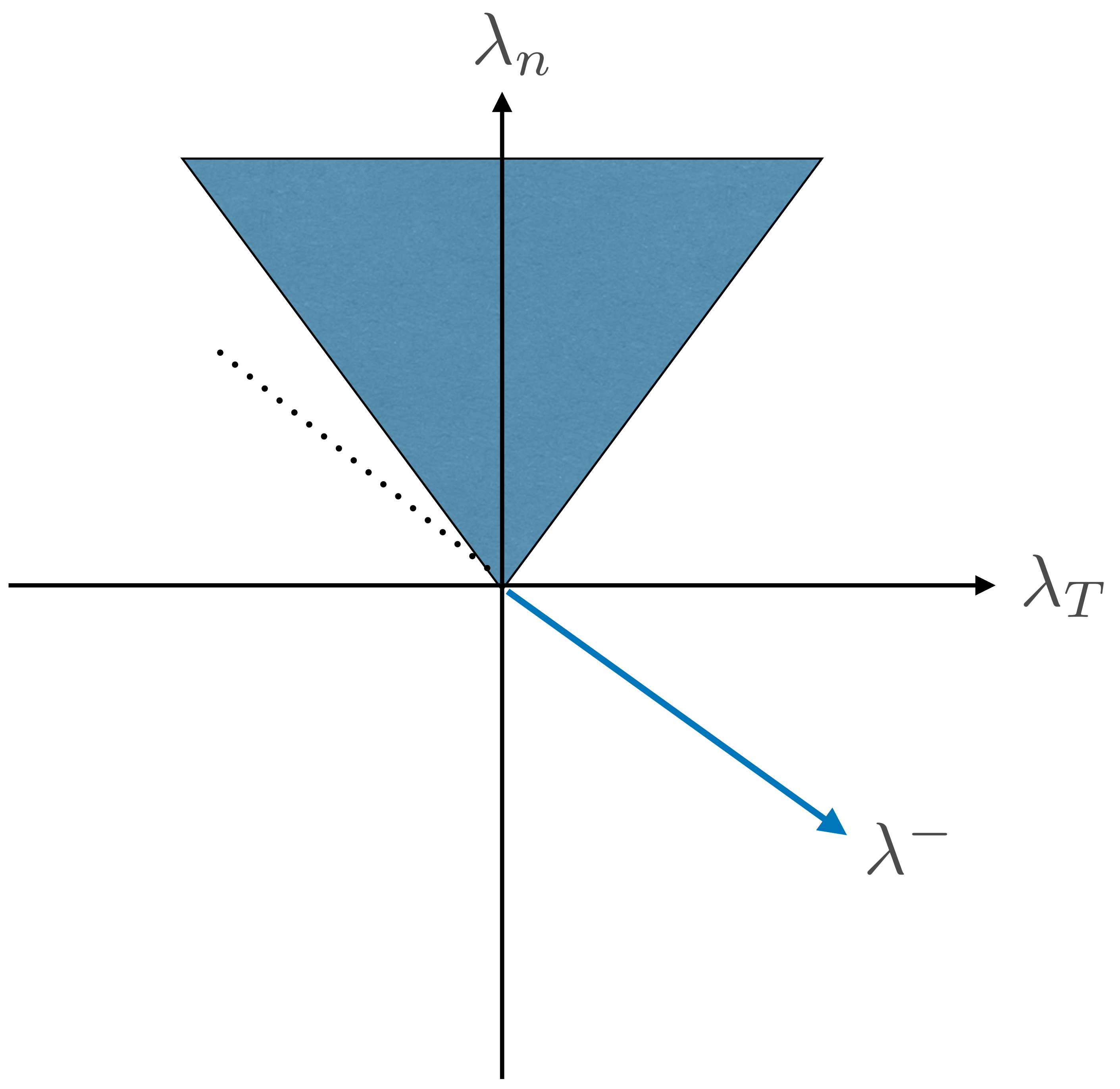
$$\mathbf{u}_\lambda = \mathbf{U}^{k+1} + \Delta t \frac{1}{m}[(\lambda_n - \lambda^-).\mathbf{n} + \lambda_\mathbf{T}]$$

$$J(\lambda^-,\lambda_n,\lambda_T) = (\lambda_n - \lambda^-)[(D^k + \Delta t(\frac{\mathbf{u}_{\lambda\mathbf{n}} + \mathbf{U}_n}{2})] + s\lambda_n + \langle \lambda_\mathbf{T}, \Delta t \frac{\mathbf{u}_{\lambda\mathbf{T}} + \mathbf{U}_T}{2})\rangle$$

$$K^k = \{(\lambda^-,\lambda_n,\lambda_T),\ \mu\lambda_n \geq |\lambda_\mathbf{t}|\ \lambda^- \geq 0,\ \lambda_n\lambda^- = 0\}$$

$$F(s) = \mu\Delta t|\mathbf{u}^\mathbf{s}_\mathbf{T}|$$

The non convex constraint




- Existence of a minimum

# Modeling Viscosity and Friction : The algorithm

The dual problem for viscosity and friction with fixed point

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$$\mathbf{u}_\lambda = \mathbf{U}^{k+1} + \Delta t \frac{1}{m}[(\lambda_n - \lambda^-).\mathbf{n} + \lambda_\mathbf{T}]$$

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$$K^k = \{(\lambda^-,\lambda_n,\lambda_T),\ \mu\lambda_n \geq |\lambda_\mathbf{t}|\ \lambda^- \geq 0,\ \lambda_n\lambda^- = 0\}$$

$$F(s) = \mu\Delta t|\mathbf{u}^\mathbf{s}_\mathbf{T}|$$

Knowing $q^n,\ u^n,\ \gamma^n$

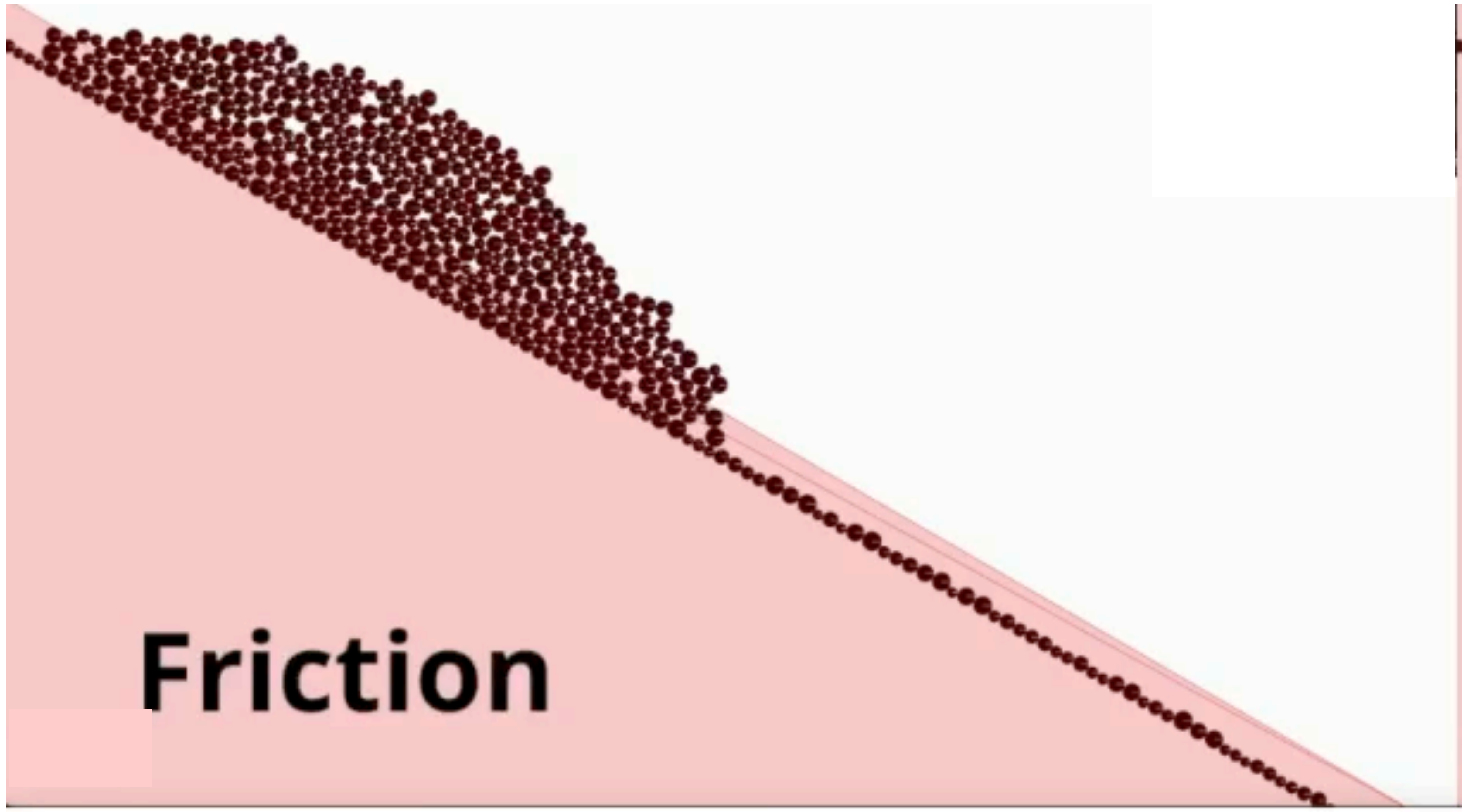
Step 1 : $\lambda^{n+1}$ sol of $\min_{\lambda\in K^n} J(-\lambda)$
with Projected Gradient algorithm
and Fixed Point

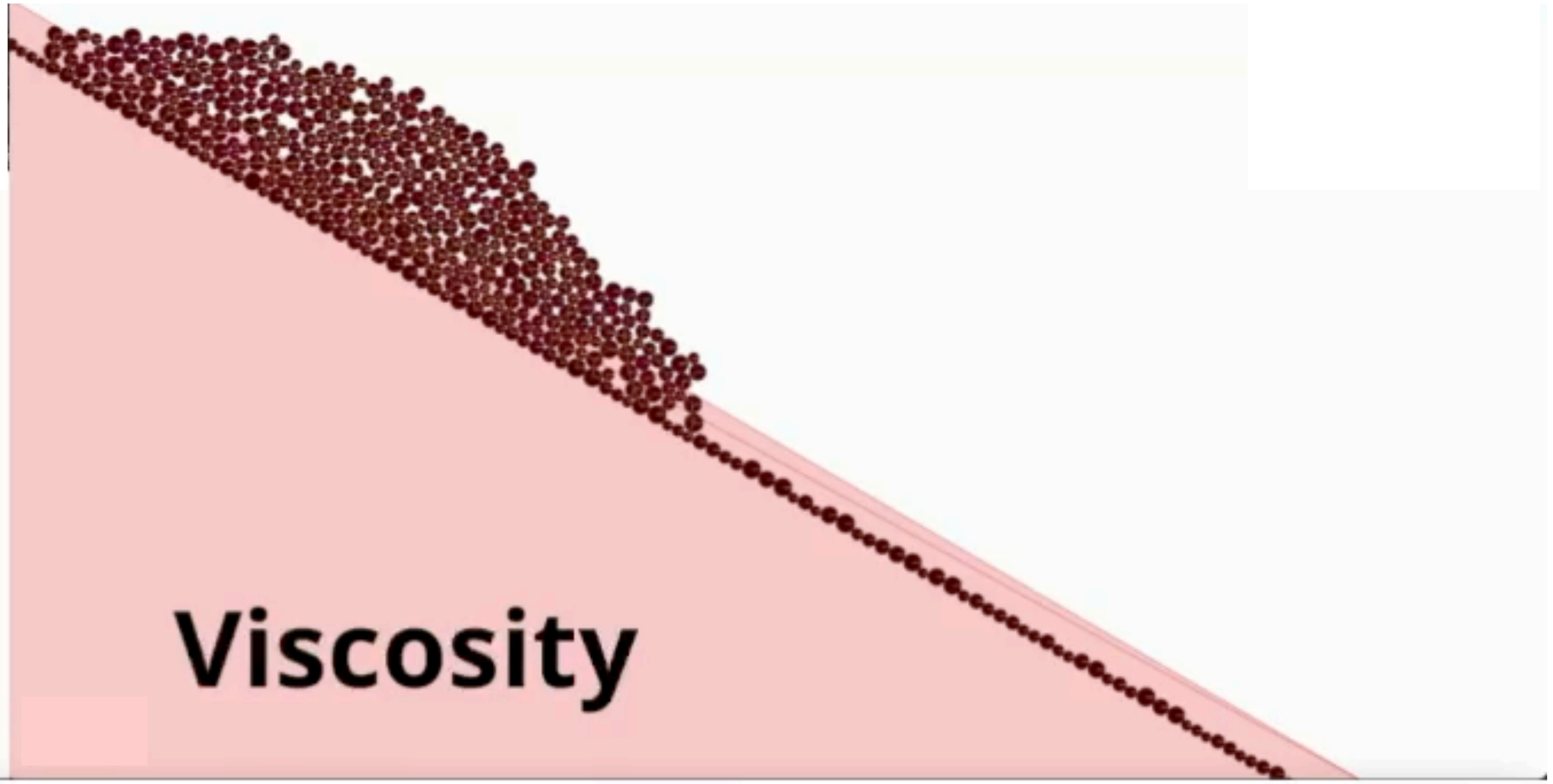
Step 2 : $u^{n+1} = U^{n+1} + \frac{dt}{m}\lambda^{n+1}$

Step 3 : $q^{n+1} = q^n + dtu^{n+1}$

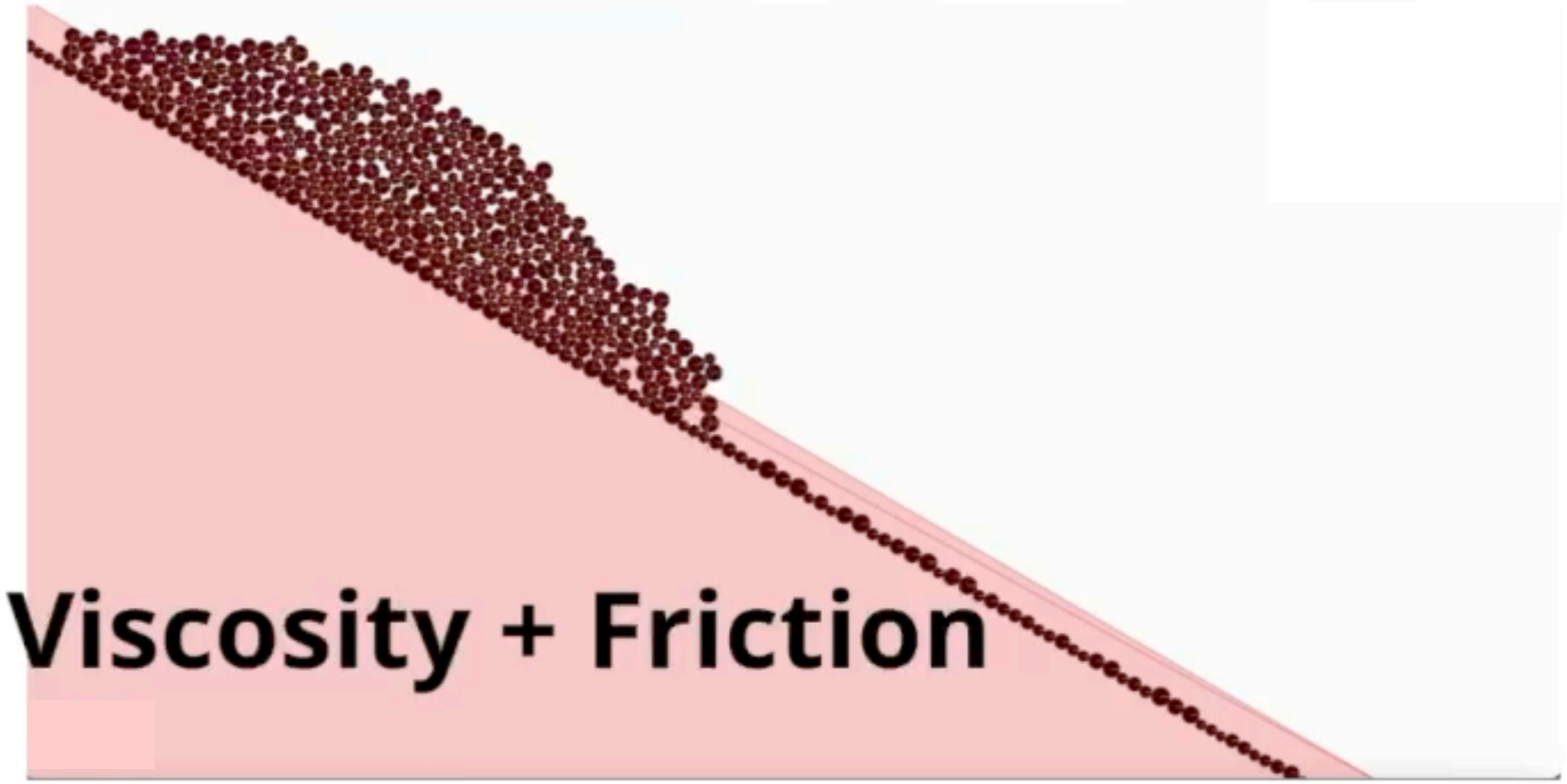
Step 4 : $\gamma^{n+1} = P_{0\geq\gamma\geq\gamma_{min}}(\gamma^n - dt\lambda^{n+1}.\mathbf{n})$

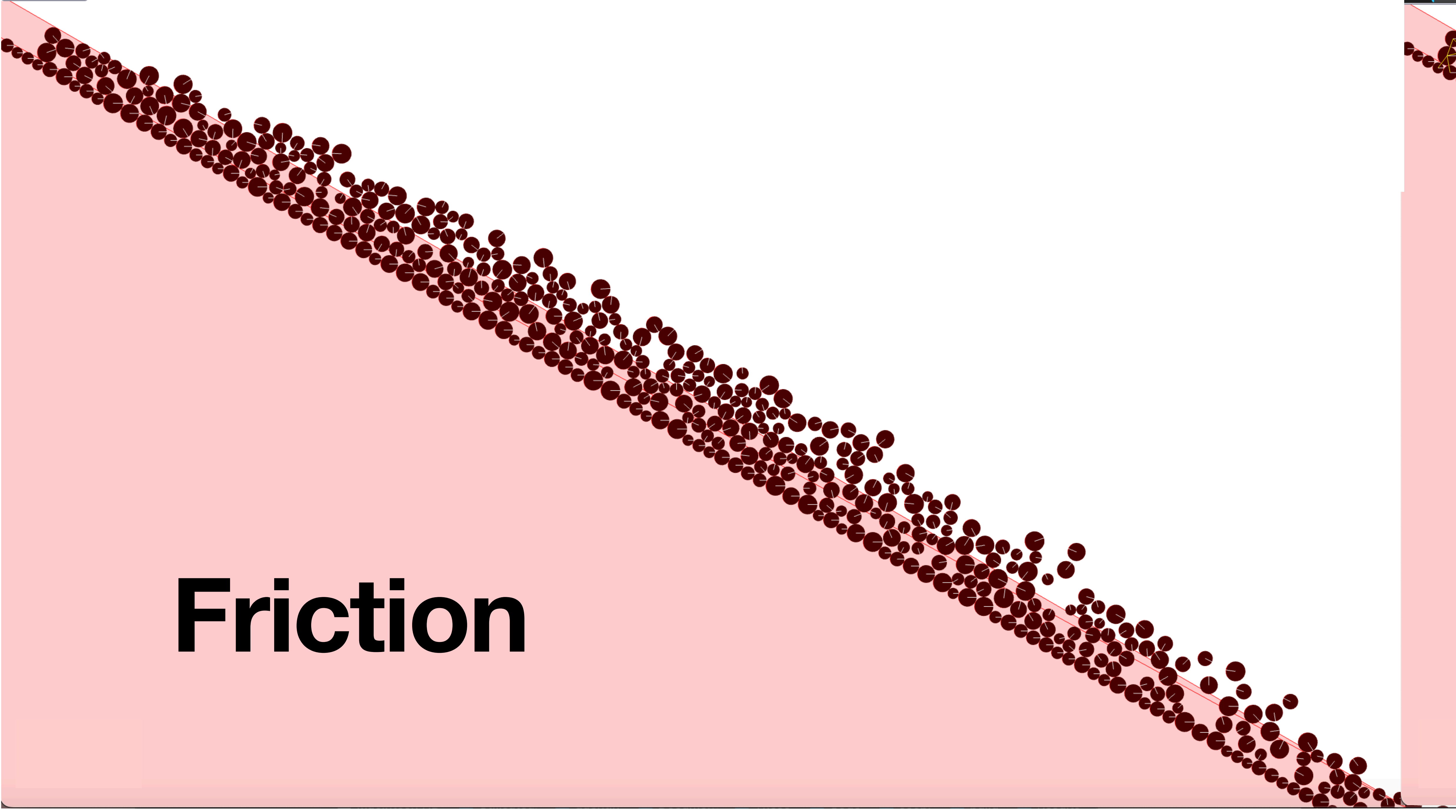
- Converge in O(dt)

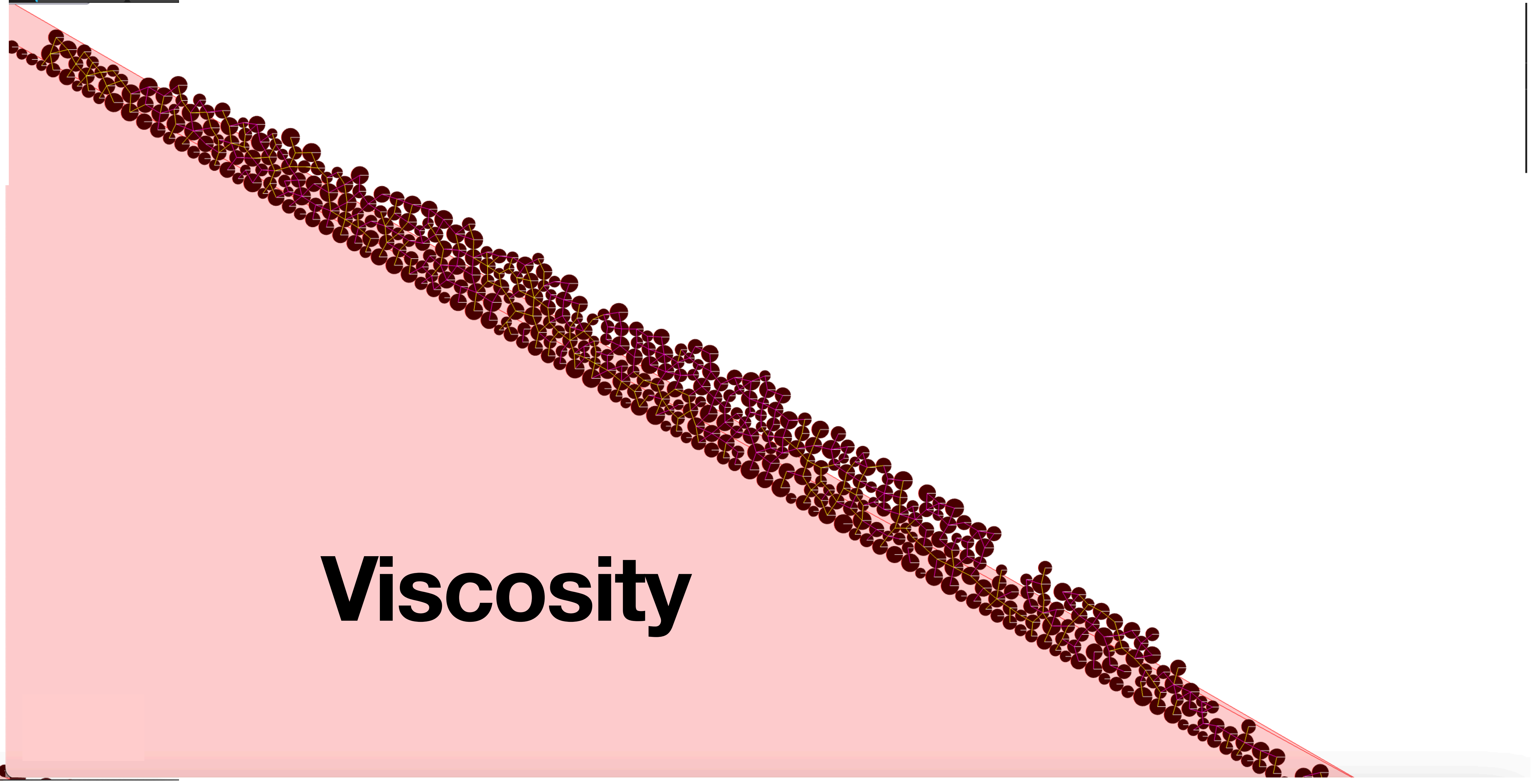



**<u>Simulations Done with Scopi</u>**



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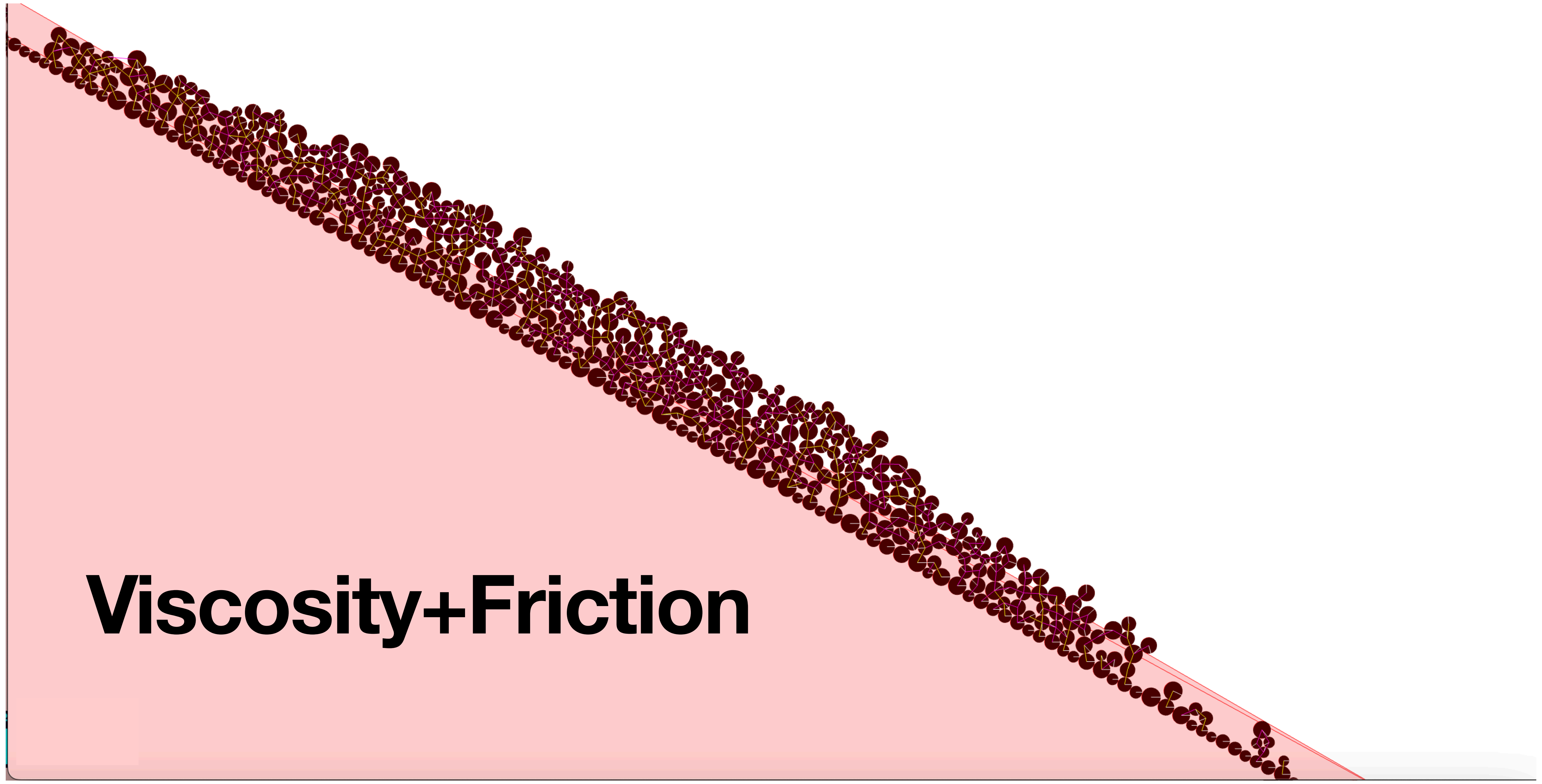

# Prospects

- Model validation by comparison with micro and macro experiments. Micro with Maxime Nicolas at IUSTI. Macro with Georges Gauthier and Baptiste Darbois-Texier at FAST

- Study of the convergence of the numerical scheme to the model

- Study of the optimization problem