Hierarchical matrix preconditioning for dominant advection problems

Arthur Saunier[*,†]

under the supervision of

A.ANCIAUX[*] L.AGÉLAS[*] I.BEN GHARBIA[*] X.CLAEYS[†]

[*]IFPEn [†]Sorbonne Université

Plan

Plan

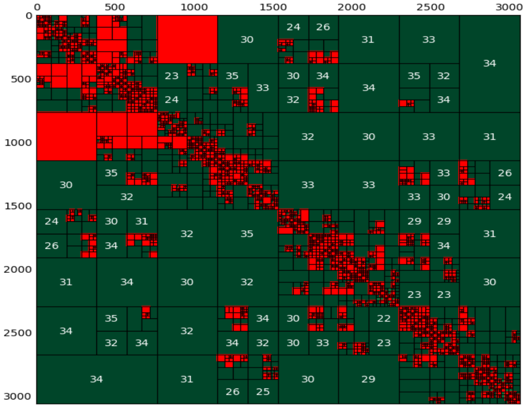
Introduction

- Advection diffusion: $-\varepsilon\Delta u + b\cdot\nabla u + cu = f$ on Ω, $b \in \mathbb{R}^3, c, f \in L^2(\Omega)$, $u|_{\partial\Omega} = 0$.
  Dominant advection :$\varepsilon \to 0$.
  - Numerical resolution of PDEs by discretization methods (FEM:finite elements methods, ...).
  - Sparse linear system $Ax = y$.
  - Numerical simulations $\to$ inversion of large sparse matrices.
- Preconditioned iterative methods: $M^{-1}Ax = M^{-1}y$.
- Choice of the preconditioner: ILU with k level of fill-in.
  - $ILU(0)$ same sparsity, convergence issue.
  - $ILU(k)$ lost of the sparsity, convergence improved with k.
- $\mathcal{H}$-matrices: approximate the inverse in quasi-linear complexity.
  - Storage of $A_{\mathcal{H}}^{-1}$: $\mathcal{O}(n\log(n))$ against $\mathcal{O}(n^2)$.
  - Factorisation $\mathcal{H}$-LU : $\mathcal{O}(n\log(n)^3)$ against $\mathcal{O}(n^3)$.

General idea : $\mathcal{H}$-matrices

- Block representation to approximate/compress a matrix.

  - Compressed blocks (approximated blocks).
  - Dense blocks (blocks left identical).



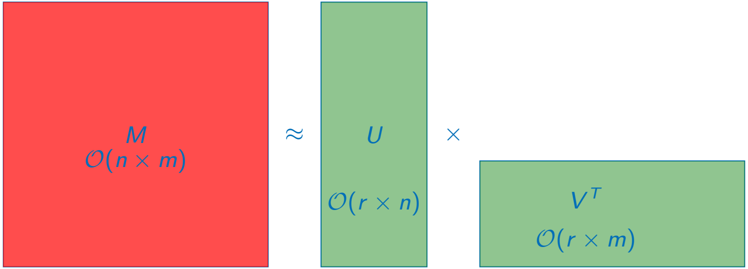
Hierarchical representation:
invert of a FEM matrix
compressed blocks, dense blocks

Plan

Two key ingredients

- Compression
  - Compression method $\rightarrow$ Low rank approximation.
- Partitioning
  - Correspondence DOF $\longleftrightarrow$ geometrical points.
  - Block representation: Splitting strategy $\rightarrow$ Cluster tree.
  - Partitioning criterion: Admissibility condition $\rightarrow$ Block cluster tree.

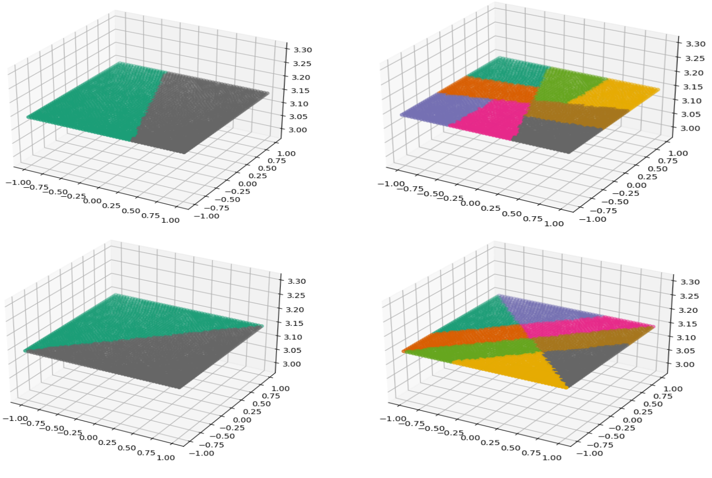
First key ingredient: Low rank approximation



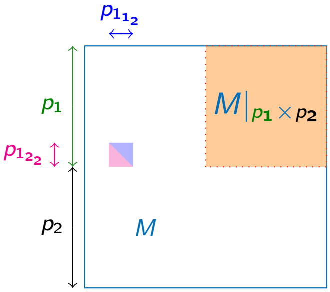
- Tolerance $tol > 0$, $||M - UV^T||_F \leq tol||M||_F$ with $r \ll n$.
- Low-rank format: Arithmetic in linear complexity.
- Best low rank approximation: truncated SVD in $\mathcal{O}(n^3)$.
- Factorisation up to rank r_0/precision tol in $\mathcal{O}(r_0 n)$ (Adaptive Cross Approximation).

Block representation and Cluster tree

Cluster tree = Recursif partitioning of the cluster of points given a splitting strategy.



$depth = 1$ $\qquad$ $depth = 3$



Interaction:

$p_2 \times p_1$ $(depth = 1)$,

$p_{1_{2_2}} \times p_{1_{1_2}}$ $(depth = 3)$.

Second key ingredient: Block Cluster Tree

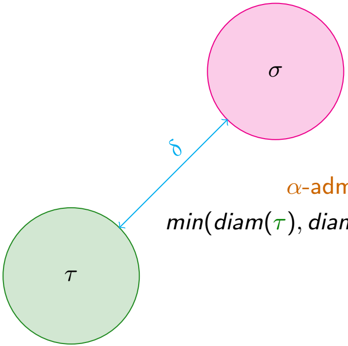
- Cluster of points $\mathcal{P}+$ Splitting strategy $\Rightarrow$ Cluster tree.

Second key ingredient: Block Cluster Tree

- Cluster of points $\mathcal{P}+$ Splitting strategy $\Rightarrow$ Cluster tree.
- Recursively build a partition of $\mathcal{P} \times \mathcal{P}$ with the cluster tree: Block cluster tree.

Second key ingredient: Block Cluster Tree

- Cluster of points $\mathcal{P}$+ Splitting strategy $\Rightarrow$ Cluster tree.
- Recursively build a partition of $\mathcal{P} \times \mathcal{P}$ with the cluster tree: Block cluster tree.
- Node $M|_{\tau\times\sigma} \leftrightarrow$ interaction between two nodes τ and σ of the cluster tree.
  - (τ, σ) admissible
    $\Rightarrow M|_{\tau\times\sigma}$ is a leaf.
  - Else
    $$M|_{\tau\times\sigma} = \begin{pmatrix} M|_{\tau_1\times\sigma_1} & M|_{\tau_1\times\sigma_2} \\ M|_{\tau_2\times\sigma_1} & M|_{\tau_2\times\sigma_2} \end{pmatrix}$$
- $\tau \times \tau$ cannot be admissible.



α-admissibility:

$$min(diam(\tau), diam(\sigma)) \leq 2\alpha \underbrace{dist(\tau, \sigma)}_{\delta}$$

$\mathcal{H}$-matrix

Recursively build the $\mathcal{H}$-matrix with the block cluster tree by approximating its leaves.

Algorithm 0: Hierarchical formatting

build-Hmatrix $(\tau \times \sigma)$:

if $\tau \times \sigma$ *is admissible* **then**

  $M_{\mathcal{H}}|_{\tau \times \sigma} = LowRank(M|_{\tau \times \sigma})$

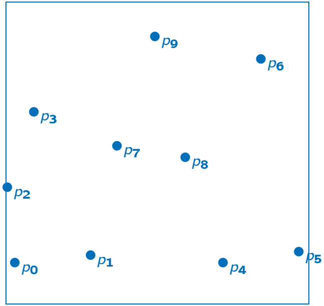
else

  for $(\tau' \in sons(\tau), \sigma' \in sons(\sigma))$ **do**

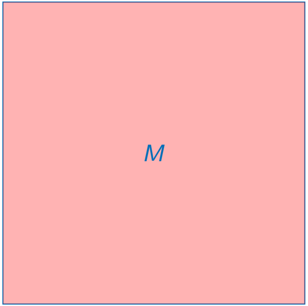
    build-Hmatrix $(\tau' \times \sigma')$

  end

end



Cluster tree
$depth = 0$



$\mathcal{H}$-matrix
$depth = 0$

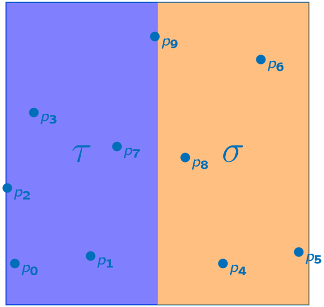
$\mathcal{H}$-matrix

Recursively build the $\mathcal{H}$-matrix with the block cluster tree by approximating its leaves.

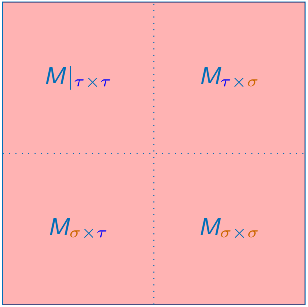
Algorithm 0: Hierarchical formatting

build-Hmatrix $(\tau \times \sigma)$:

if $\tau \times \sigma$ *is admissible* **then**
  $M_{\mathcal{H}}|_{\tau\times\sigma} = LowRank(M|_{\tau\times\sigma})$
else
  for $(\tau' \in sons(\tau), \sigma' \in sons(\sigma))$ **do**
    build-Hmatrix $(\tau' \times \sigma')$
  end
end



Cluster tree
$depth = 1$



Block matrix
$depth = 1$

$\mathcal{H}$-matrix

Recursively build the $\mathcal{H}$-matrix with the block cluster tree by approximating its leaves.

Algorithm 0: Hierarchical formatting

build-Hmatrix $(\tau \times \sigma)$:

if $\tau \times \sigma$ *is admissible* **then**

  $M_{\mathcal{H}}|_{\tau\times\sigma} = LowRank(M|_{\tau\times\sigma})$

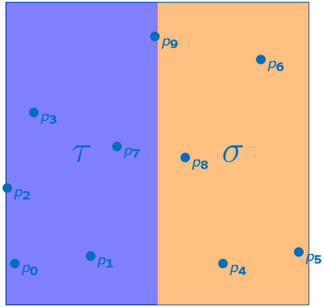
else

  for $(\tau' \in sons(\tau), \sigma' \in sons(\sigma))$ **do**

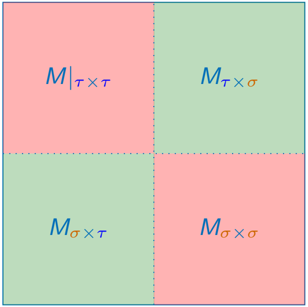
    build-Hmatrix $(\tau' \times \sigma')$

  end

end



Cluster tree
depth $= 1$



$\mathcal{H}$-matrix
depth $= 1$

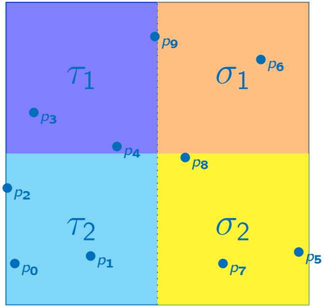
$\mathcal{H}$-matrix

Recursively build the $\mathcal{H}$-matrix with the block cluster tree by approximating its leaves.

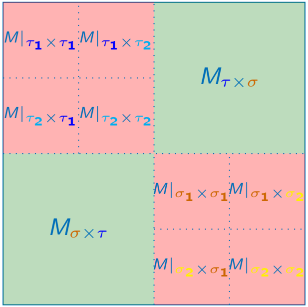
Algorithm 0: Hierarchical formatting

build-Hmatrix $(\tau \times \sigma)$:

if $\tau \times \sigma$ *is admissible* **then**
  $M_{\mathcal{H}}|_{\tau \times \sigma} = LowRank(M|_{\tau \times \sigma})$
else
  for $(\tau' \in sons(\tau), \sigma' \in sons(\sigma))$ **do**
    build-Hmatrix $(\tau' \times \sigma')$
  end
end



Cluster tree
$depth = 2$



Block matrix
$depth = 2$

$\mathcal{H}$-matrix

Recursively build the $\mathcal{H}$-matrix with the block cluster tree by approximating its leaves.

Algorithm 0: Hierarchical formatting

build-Hmatrix ($\tau \times \sigma$):

if $\tau \times \sigma$ *is admissible* **then**

  $M_{\mathcal{H}}|_{\tau \times \sigma} = LowRank(M|_{\tau \times \sigma})$

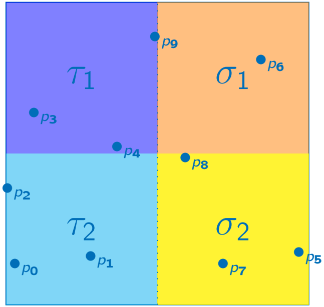
else

  for $(\tau' \in sons(\tau), \sigma' \in sons(\sigma))$ **do**

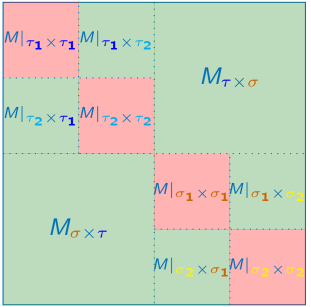
    build-Hmatrix ($\tau' \times \sigma'$)

  end

end



Cluster tree
$depth = 2$



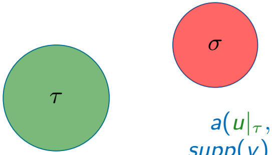
$\mathcal{H}$-matrix
$depth = 2$

Plan

Theoretical study: $-\varepsilon\Delta u + b\cdot\nabla u + cu = f$

- Solution u with $u = 0$ on $\partial\Omega$, $b \in \mathbb{R}^3$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ and $\varepsilon > 0$, **dominant advection:** $\varepsilon \to 0$.
- Variational formulation: $a(u,v) = \int_\Omega \varepsilon\nabla u\cdot\nabla v + b\cdot\nabla u\, v + cu\, dx = \int_\Omega fv\, dx \quad \forall v \in H_0^1(\Omega)$
- Discrete problem $AU = F$ (example: finite elements methods).
- Approximate A^{-1} with $\mathcal{H}$-matrix?
  - Elliptic problems: [W.Hackbush], [M.Bebendorf].
- Compression really low with the $\mathcal{H}$-matrix format for $\varepsilon \to 0$.
  - Dominant advection: [S.Le Borne] heuristic reasoning on structured meshes.
  - Extend the proofs of [M.Faustmann, J.Melenk, D.Praetorious] and [S.Börm] to the case $\varepsilon \to 0$.

Idea: Use the "orthogonality" of a:



$a(u|_\tau, v) = \int_\Omega f v = 0$
$supp(v) \subset \tau, supp(f) \subset \sigma$

Separation of the support
$\Rightarrow$ "orthogonality"

How it works: admissibility condition $\Rightarrow$ $\mathcal{H}$-matrix approximability

- What admissibility condition on (τ, σ) would ensure a good hierarchical approximation ?
- **Corollary:** Assume (τ, σ) admissible, then $\exists q \in (0,1)$, C, $C_{dim} > 0$ such that $\forall p \in \mathbb{N}_{\geq 2}$ we can find $U \in \mathbb{R}^{|\tau| \times k}$, $V \in \mathbb{R}^{|\sigma| \times k}$ (with $k \leq C_{dim} p^{dim(\Omega)+1}$) satisfying $\forall x \in \mathbb{R}^{|\sigma|}$

$$||(A^{-1}|_{\tau \times \sigma} - UV^T)x||_{Frobenius} \leq Cq^p ||x||$$

- **Theorem**: Assume (τ, σ) admissible, then $\exists q \in (0,1)$, $p \in \mathbb{N}^2$, $C, C_1, C_2 > 0$ and $v \in V$, where $dim(V) < Cp^{dim(\Omega)+1}$, locally approximating the solution on τ with the estimates

$$||\nabla(u|_\tau - v)||_{L^2(\tau)} \leq C_1 q^p ||f||_{L^2(\Omega)}$$
$$||u|_\tau - v||_{L^2(\tau)} \leq C_2 q^{p+1} ||f||_{L^2(\Omega)}$$

- Idea of the proof: Successive approximations on narrower sets.

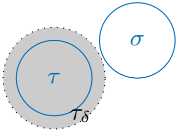
Key ingredient of the proof: Caccioppoli inequality

- $||\nabla u||_{L^2(\tau)} \leq \frac{C}{dist(\tau,\partial\tau^+)}||u||_{L^2(\tau^+)}$ where $\tau \subsetneq \tau^+$.
- "orthogonality" of a for two clusters τ and σ with $dist(\tau,\sigma)=\delta>0$ and $supp(f)\subset\sigma$.

  Separation $\Rightarrow \tau_\delta = \{x\in\Omega | dist(x,\tau)<\delta\}$.

$\forall\lambda>1$, $\exists\eta$ with $supp(\eta)\subset\tau_\delta$, $\eta|_\tau = 1$ and

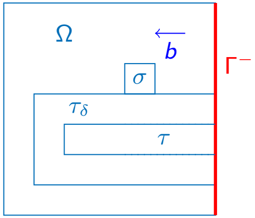
$||\nabla\eta||_{L^\infty(\Omega)} \leq \frac{\lambda}{\delta}$



Support of the cut-off η

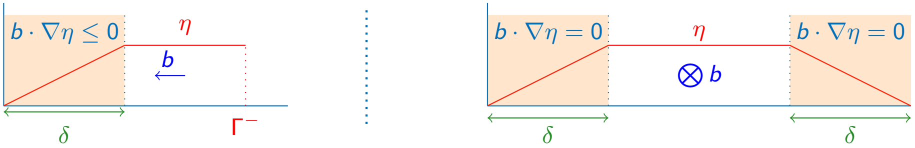
$$
\begin{aligned}
||\nabla u||^2_{L^2(\tau)} &\leq ||\nabla(\eta u)||^2_{L^2(\tau_\delta)} \\
&= <\nabla u, \nabla(\eta^2 u)>_{L^2(\Omega)} + <u\nabla\eta, u\nabla\eta>_{L^2(\Omega)} \\
&= \varepsilon^{-1}\, a(u,\eta^2 u) \;(=0)\; - \varepsilon^{-1}\int_\Omega b\cdot\nabla(u)\eta^2 u\,dx \\
&\quad - \varepsilon^{-1}\int_\Omega c\eta^2 u^2\,dx + <u\nabla\eta, u\nabla\eta>_{L^2(\Omega)} \\
&\leq \frac{\lambda^2}{\delta^2}||u||^2_{L^2(\tau_\delta)} + \int_\Omega \eta u^2 \frac{b}{\varepsilon}\cdot\nabla\eta\,dx.
\end{aligned}
$$

Choice of the geometry: vanishing of the term $b \cdot \nabla\eta$



Support of the cut-off in τ_δ.

Cluster τ aligned on advection streams and which reach the incident border $\Gamma^- = \{\mathbf{n}_{\partial\Omega} \cdot b < 0\}$.



Variation of η in the directions b and $b^\perp$.

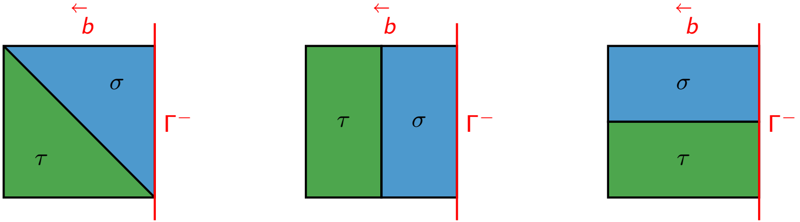
Admissible partitionning

- Caccioppoli $\Rightarrow$ α-admissibility, explicit admissibility condition:

$$(\tau, \sigma) \text{ is admissible if } \tau \text{ is aligned with } b \text{ which reaches } \Gamma^{-} \textit{ and}$$

$$dist(\tau, \sigma) > 2\alpha diam_{\infty}(\tau)$$

- Clustering strategy $\rightarrow$ suited partitioning for the Caccioppoli frame.

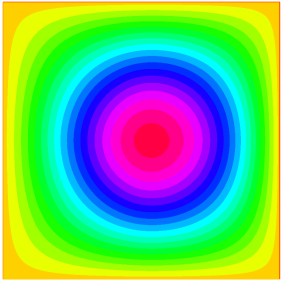


Not suited partitionning	ill suited partitionning	Well suited partitionning
τ and σ not in the frame	τ not in the frame	τ and σ in the frame

Test case: matrices obtained with FreeFem

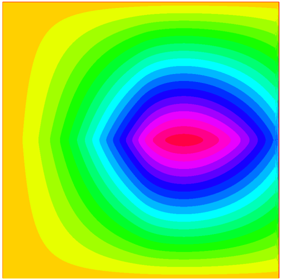
On $[0,1] \times [0,1]$, with an unstructured triangular mesh of 3057 nodes:

$$-\varepsilon \Delta u + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \nabla u + 2u = 10(1-x)(1-y)$$



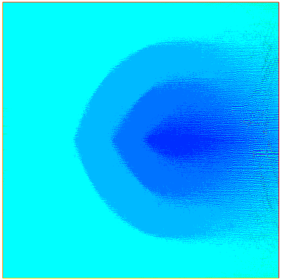
$\varepsilon = 1$

Advection diffusion



$\varepsilon = 10^{-3}$

Transitory case



$\varepsilon = 10^{-6}$

Dominant advection

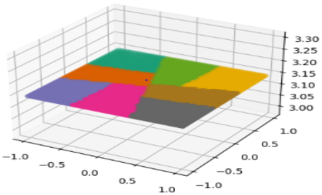
Approximation of A^{-1}

- C++ library HTOOL for $\mathcal{H}$-matrices (Pierre Marchand INRIA).
- Standard approach: Principal Component Analysis + α-admissibility.
- New approach: Partitioning into tubes aligned with the convection + new condition.
- $M_{\mathcal{H}}$ hierarchical approximation of M, $\mathcal{F} = \{\text{leaves of } M_{\mathcal{H}}\}$, $\mathcal{R} = \{\text{admissible leaves of } M_{\mathcal{H}}\}$,

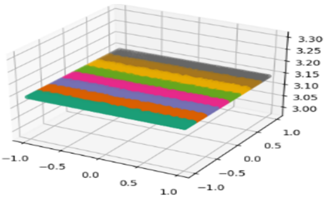
$$Compression(M_{\mathcal{H}}) = \frac{\sum_{R\in\mathcal{R}} \text{rank}(R)(\text{line}(R) + \text{col}(R)) + \sum_{F\in\mathcal{F}\setminus\mathcal{R}} \text{line}(F) \times \text{col}(F)}{\text{line}(M) \times \text{col}(M)}$$

$$error(M_{\mathcal{H}}) = \frac{||M - M_{\mathcal{H}}||_F}{||M||_F}$$

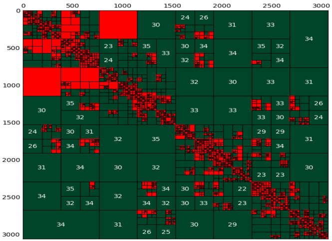
Approximation of A^{-1}: C++ library HTOOL



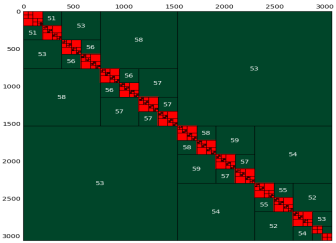
Splitting PCA



New splitting

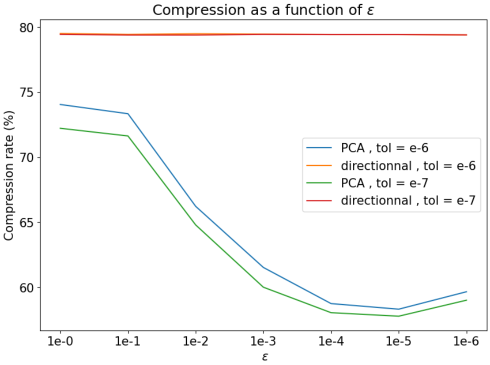


Ususual condition + PCA
(State of the art)

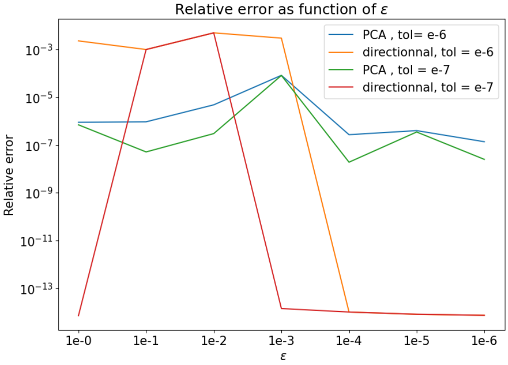


New condition + new splitting
(Our method)

Quality of approximations: matrices 3057 × 3057



Compression VS ε

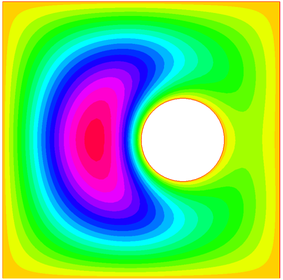


Relative error VS of ε

Compression+20%, error 10^9 times lower!

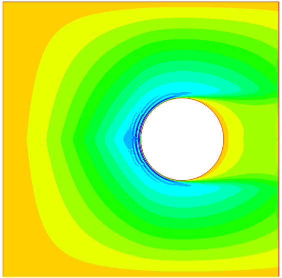
Test case: domain with a hole

The same equation on a domain with a hole, mesh with3227 nodes.



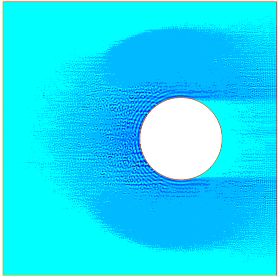
$\varepsilon = 1$

Advection diffusion



$\varepsilon = 10^{-3}$

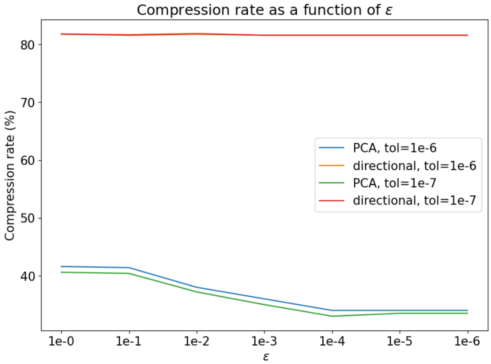
Transitory case



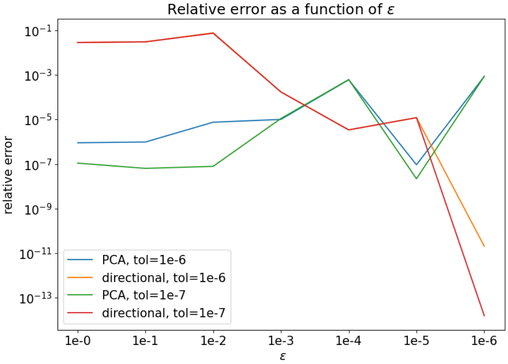
$\varepsilon = 10^{-6}$

Dominant advection

Quality of approximations: matrices 3227×3227



Compression VS ε



Relative error VS ε

Compression +40% and error less than 10^{-5} for $\varepsilon \leq 10^{-4}$.

Plan

Conclusion

- Proposal for a new admissibility condition and a new partitioning suited for dominant advection.
- The compression and error results obtained in our tests significantly surpass those of the current state of the art.

THANK YOU!

[1] M.Bebendorf. *Hierarchical matrices : a means to efficiently solve elliptic boundary value problems*, volume 63 of *Lecture notes in computational science and engineering*. 2008.

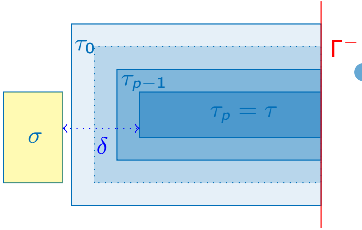
[2] L.Grasedyck W.Hackbusch and S.Börm. An introduction to hierarchical matrices, 2001.

[3] S.Börm. *Efficient numerical methods for non-local operators: H2-matrix compression, algorithms and analysis*, volume 14. European Mathematical Society, 2010.

[4] J.Melenk M.Faustmann and D.Praetorius. $\mathcal{H}$-matrix approximability of the inverses of FEM matrices. *Numerische Mathematik*, 131(4), 2015.

[5] S.Le Borne and L.Grasedyck. H-matrix preconditioners in convection-dominated problems. *SIAM J. Matrix Anal. Appl.*, 27(4), 2006.

Local approximation in low dimension



Concentric bounding boxes

- We impose $diam_\infty(\tau) \leq 2\alpha\delta$, $\delta > dist(\tau_0, \sigma) = \gamma\delta > 0$
- Poincaré Wirtinger: $v_1 \in V_1$, $dim(V_1) = l^{dim(\Omega)}$, $l \in \mathbb{N}$ with $||u - v_1||_{L^2(\tau_0)} \leq \frac{diam_\infty(\tau_0)}{l} ||\nabla u||_{L^2(\tau_0)}$.
- Cacciopoli : $\tau_1 \to \tau_0$

$$||\nabla(u - v_1)||_{L^2(\tau_1)} \leq \frac{C}{\text{dist}(\tau_1, \partial\tau_0)} ||u - v_1||_{L^2(\tau_0)}$$

$$\leq C' \underbrace{\frac{p}{l(1-\gamma)}}_{q<1} ||\nabla u||_{L^2(\tau_0)}$$

PW +condition

- $\forall k \leq p$ we find $v_k \in V_k$ approximation of $u - \sum_{i \leq k-1} v_i$ in $L^2(\tau_{k-1})$ (PW) then in $H^1(\tau_k)$ (Cacciopoli).

Local approximation in low dimension

- $\tau_p = \tau,\ v = \sum_{i\leq p} v_i|_\tau \in V = \sum_{i\leq p} V_i$ where $dim(V) \leq pl^d$ and we have the estimates:

$$||\nabla(u-v)||_{L^2(\tau)} \leq Cq^p||\nabla u||_{L^2(\tau_0)} \lesssim^? Cq^p||f||_{L^2(\Omega)}$$

$$||u-v||_{L^2(\tau)} \leq C'q^{p+1}||\nabla u||_{L^2(\tau_0)} \lesssim^? C'q^{p+1}||f||_{L^2(\Omega)}$$

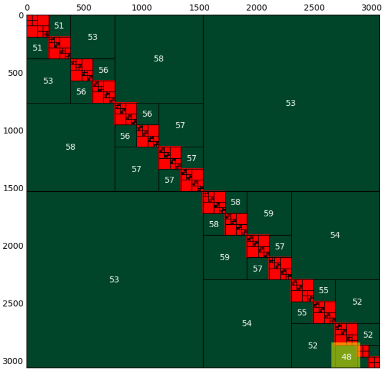
Bound independent of u

- $\text{dist}(\tau_0, \partial\tau_\delta) = \gamma\delta > 0 \Rightarrow$ Cacciopoli: $||\nabla u||_{L^2(\tau_0)} \leq \frac{C}{\gamma\delta}||u||_{L^2(\tau_\delta)}$.
- $c_0 = \inf(c - \text{div}(\frac{b}{2})) > 0 \Rightarrow ||u||_{L^2(\Omega)}||f||_{L^2(\Omega)} \geq a(u,u) \geq c_0||u||^2_{L^2(\Omega)}$.
- QED and explicit admissibility condition (τ, σ):

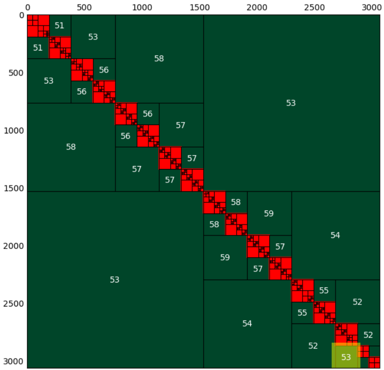
The couple(τ,σ) is said to be admissible if τ is an aligned cluster on b reaching Γ^-

and $\text{dist}(\tau,\sigma) > 2\alpha\text{diam}_\infty(\tau)$

Importance of the tolerance of ACA

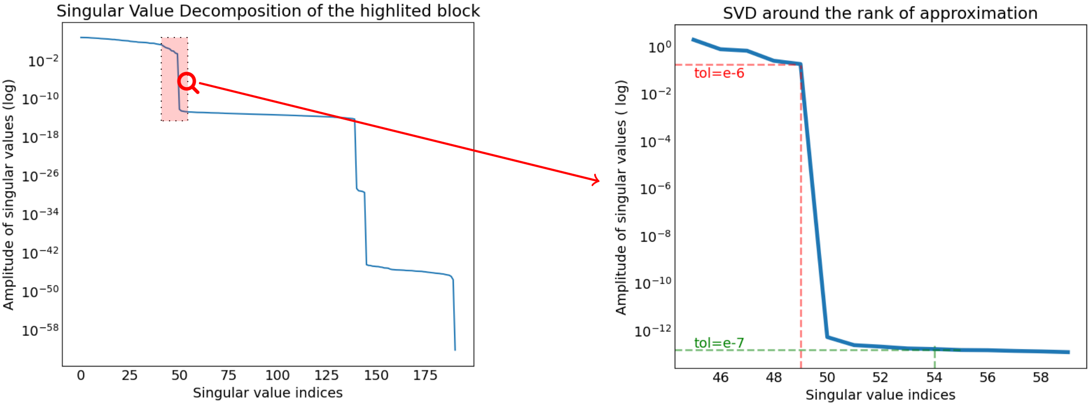


tolerance $= 10^{-6}, \varepsilon = 10^{-3}$



tolerance $= 10^{-7}, \varepsilon = 10^{-3}$

Shape of the block's SVD



Error of a rank k approximation is proportional to the $k+1^{th}$ singular value.