Weighted and uniform optimal control of ensembles via Γ-convergence

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Ensembles of control systems: motivations

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Let us consider the following model for chemotherapy:

$$\begin{cases} \dot{x}_1 = \xi_1 x_1 \left(1 - \frac{x_1 + x_2}{M}\right) - \mu u x_1 & \text{(sensitive population)} \\ \dot{x}_2 = \xi_2 x_2 \left(1 - \frac{x_1 + x_2}{M}\right) & \text{(resistant population)} \end{cases}$$

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Classical strategy in medicine

Maximal dose: $u(t) \equiv u_{\max}$ until the tumor starts growing again.

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Classical strategy in medicine

Maximal dose: $u(t) \equiv u_{\max}$ until the tumor starts growing again.
Then, when possible, change drug (2nd line treatment) and use it at the maximal dose.

Ensembles of control systems: motivations

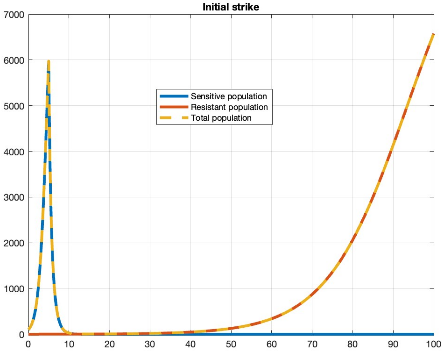

Figure: Strategy $u(t) \equiv u_{\max}$. The sensitive population is rapidly extincted by the treatment. After some time, a resistant tumor returns.

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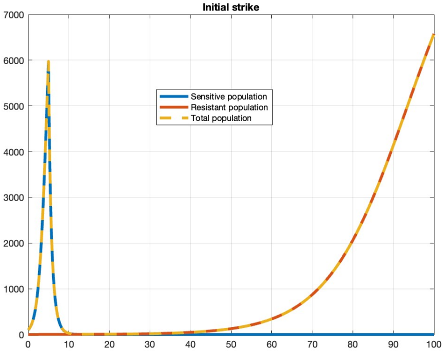

Figure: Strategy $u(t) \equiv u_{\max}$. The sensitive population is rapidly extincted by the treatment. After some time, a resistant tumor returns.

This strategy does not require the knowledge of ξ_1, ξ_2, M, μ.

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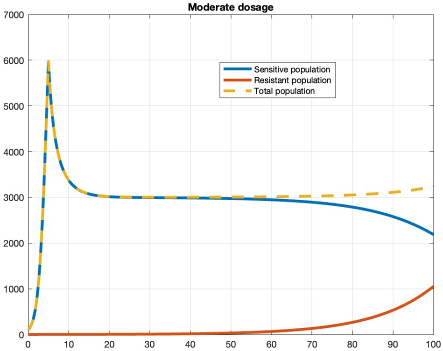

Figure: Strategy $u(t) \equiv \bar{u} < u_{\max}$. The tumor never disappears, but it is stabilized. The sensitive cells are delaying the growth of the resistant population.

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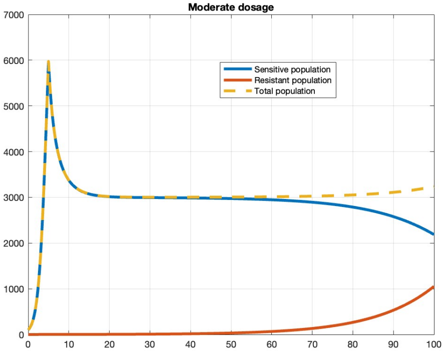

Figure: Strategy $u(t) \equiv \bar{u} < u_{\max}$. The tumor never disappears, but it is stabilized. The sensitive cells are delaying the growth of the resistant population.

This strategy **depends on** ξ_1, ξ_2, M, μ.

Ensembles of control systems: motivations

Control of a qubit:

$$\frac{d\psi}{dt} = \begin{pmatrix} E + \alpha & u(t) \\ u(t) & -E - \alpha \end{pmatrix} \psi,$$

where α represents the uncertainty affecting the *resonance frequency*.

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In [Robin, Augier, et al., J.Diff.Eq., 2022] a strategy for *uniform ensemble controllability* is proposed.

Is it possible to find *optimal strategies*?
Optimal could be *on average* on the ensemble, or *uniformly*.

Ensembles of affine-control systems

Given a compact set of parameters $\Theta \subset \mathbb{R}^d$, we consider a family of affine-control systems in $\mathbb{R}^n$ on the time interval $[0, T]$:

$$\dot{x}^\theta = b^\theta(x) + A^\theta(x)u, \quad x^\theta(0) = x_0^\theta,$$

simultaneously controlled by $u \in \mathcal{U} := L^p([0, T], \mathbb{R}^m)$, $1 < p < \infty$.

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Modelling data uncertainty affecting the dynamics or the initial datum.

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Technical assumptions

- $(x, \theta) \mapsto b^\theta(x) \in \mathbb{R}^n$, $(x, \theta) \mapsto A^\theta(x) \in \mathbb{R}^{n \times m}$ Lipschitz-continuous;
- $\theta \mapsto x_0^\theta$ is Lipschitz-continuous.

Optimal control of ensembles: weighted problems

For every $\theta \in \Theta$, we would like to solve

$$a^\theta(x^\theta(T)) + \beta \int_0^T f(u(s))\, ds \to \min,$$

with $\beta > 0$, and where

- $a^\theta : \mathbb{R}^n \to \mathbb{R}_+$ is the end-point cost ($a : \mathbb{R}^n \times \Theta \to \mathbb{R}_+$ continuous);
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Idea

We use a probability measure $\mu \in \mathcal{P}(\Theta)$ to *quantify the uncertainty on* θ. Then, we consider

$$\mathcal{G}_\mu(u) = \int_\Theta a(x^\theta(T), \theta)\, d\mu(\theta) + \beta \int_0^T f(u(s))\, ds \to \min .$$

Existence of minimizers

Let us consider

$$\mathcal{G}_\mu(u) = \int_\Theta a(x^\theta(T), \theta)\, d\mu(\theta) + \beta \int_0^T f(u(s))\, ds.$$

Proposition

There exists $\hat{u} \in \mathcal{U}$ such that

$$\mathcal{G}_\mu(\hat{u}) = \inf_{\mathcal{U}} \mathcal{G}_\mu.$$

Moreover, for every $\hat{u} \in \arg\min \mathcal{G}_\mu$, we have $\|\hat{u}\|_{L^p} \leq C(\beta)$.

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In practice, we may have access to empirical measurements $\theta^1, \dots, \theta^M \in \Theta$, independently sampled from μ.
We define

$$\mu^M := \frac{1}{M} \sum_{j=1}^{M} \delta_{\theta^j}$$

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and we assume that

$$\lim_{M\to\infty} \langle \mu^M, \phi \rangle = \langle \mu, \phi \rangle, \quad \forall \phi \in C_b(\Theta),$$

i.e., $\mu^M \rightharpoonup^* \mu$ as $M \to \infty$.

Reduction to finite ensembles: Γ-convergence

For every $M \geq 1$, we consider $\mathcal{G}_{\mu^M} : \mathcal{U} \to \mathbb{R}$ defined as

$$\mathcal{G}_{\mu^M}(u) = \int_\Theta a(x^\theta(T), \theta)\, d\mu^M(\theta) + \beta \int_0^T f(u(s))\, ds.$$

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Theorem

For every $M \geq 1$, let us consider $\mathcal{G}_{\mu^M} : \mathcal{U} \to \mathbb{R}$.
Then, the sequence $(\mathcal{G}_{\mu^M})_{M \geq 1}$ is Γ-convergent to the functional $\mathcal{G}_\mu : \mathcal{U} \to \mathbb{R}$ with respect to the weak topology of $\mathcal{U}$.

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Remark

Here the fact that the systems are affine in the control is crucial!

Γ-convergence: consequences

Convergence of minima.

$$\min_{u \in \mathcal{U}} \mathcal{G}_{\mu^M} \to \min_{u \in \mathcal{U}} \mathcal{G}_{\mu} \quad \text{as } M \to \infty.$$

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Convergence of minimizers. Let $\hat{u}^M \in \arg\min_{\mathcal{U}} \mathcal{G}_{\mu^M}$.
Then $(\hat{u}^M)$ is pre-compact in the **weak topology** of L^p, and clusters are minimizers of $\mathcal{G}_{\mu}$.

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Convergence of integral costs. Let $\hat{u}^M \in \arg\min_{\mathcal{U}} \mathcal{G}_{\mu^M}$, and assume that $\hat{u}^M \rightharpoonup \hat{u}$. Then,

$$\lim_{M\to\infty} \int_\Theta a(x^\theta_{\hat{u}^M}(T), \theta)\, d\mu^M(\theta) = \int_\Theta a(x^\theta_{\hat{u}}(T), \theta)\, d\mu(\theta),$$

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Convergence of minimizers. Let $\hat{u}^M \in \arg\min_{\mathcal{U}} \mathcal{G}_{\mu^M}$.
Then, if f is **strictly convex**, $(\hat{u}^M)$ is pre-compact in the **strong topology** of L^p, and clusters are minimizers of $\mathcal{G}_\mu$.

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Γ-convergence and PMP: preliminaries

Idea

Try to use Γ-convergence to establish PMP for infinite ensembles.
When considering μ^M, the problem reduces to a control system in $(\mathbb{R}^n)^M$.

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For every $u \in \mathcal{U}$, we define $X_u : [0, T] \times \Theta \to \mathbb{R}^n$ as

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and $\Lambda_u : [0, T] \times \Theta \to (\mathbb{R}^n)^*$ as

$$\Lambda_u(t, \theta) := \lambda_u^\theta(t),$$

where

$$\dot{\lambda}_u^\theta = -\lambda_u^\theta \cdot \frac{\partial}{\partial x}\left(b^\theta(x_u) - A^\theta(x_u)u\right), \quad \lambda_u^\theta(T) = -\nabla_x a(x_u(T), \theta).$$

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Theorem

Let $\hat{u}^M \in \arg\min \mathcal{G}_{\mu^M}$. *Then, considering* $X_{\hat{u}^M} : [0, T] \times \Theta \to \mathbb{R}^n$ *and* $\Lambda_{\hat{u}^M} : [0, T] \times \Theta \to (\mathbb{R}^n)^*$ *as before, we have that*

$$\hat{u}^M(t) \in \arg\max_{v \in \mathbb{R}^m} \left\{ \int_\Theta \Lambda_{\hat{u}^M}(t, \theta) \cdot A^\theta\big(X_{\hat{u}^M}(t, \theta)\big) v \, d\mu^M(\theta) - \beta f(v) \right\}$$

for a.e. $t \in [0, T]$.

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for a.e. $t \in [0, T]$.

Remark

No need for computing $X_{\hat{u}^M}(t,\theta), \Lambda_{\hat{u}^M}(t,\theta)$ for every $\theta \in \Theta$.
Sufficient for $\theta \in \mathrm{supp}(\mu^M)$.

Γ-convergence and PMP: idea

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$$\begin{cases} \mu^M \rightharpoonup^* \mu \\ X_{\hat{u}^M} \to_{C^0} X_{\hat{u}} \\ \Lambda_{\hat{u}^M} \to_{C^0} \Lambda_{\hat{u}} \\ \hat{u}^M(t) \to \hat{u}(t) \quad \text{a.e. (up to subseq.)} \end{cases} \qquad \text{as } M \to \infty.$$

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We can pass to the limit here:

$$\hat{u}^M(t) \in \arg\max_{v \in \mathbb{R}^m} \left\{ \int_\Theta \Lambda_{\hat{u}^M}(t,\theta) \cdot A^\theta\big(X_{\hat{u}^M}(t,\theta)\big) v \, d\mu^M(\theta) - \beta f(v) \right\}$$

We use that the subdifferential ∂f has closed graph.

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We get:

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Γ-convergence and PMP: infinite ensembles

We have obtained:

Theorem (S., 2023)

Let $\hat{u} \in \arg\min \mathcal{G}_\mu$. *Then, considering* $X_{\hat{u}} : [0, T] \times \Theta \to \mathbb{R}^n$ *and* $\Lambda_{\hat{u}} : [0, T] \times \Theta \to (\mathbb{R}^n)^*$ *as before, we have that*

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for a.e. $t \in [0, T]$.

Optimal control of ensembles: minimax problems

For every $\theta \in \Theta$ we would like to solve

$$a^\theta(x^\theta(T)) + \beta \int_0^T f(u(s))\, ds \to \min,$$

with $\beta > 0$, and where

- $a^\theta : \mathbb{R}^n \to \mathbb{R}_+$ is the end-point cost ($a : \mathbb{R}^n \times \Theta \to \mathbb{R}_+$ continuous);
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Idea

We want to do *the best in the worst scenario*:

$$\mathcal{F}_\Theta(u) = \sup_{\theta \in \Theta} a(x^\theta(T), \theta) + \beta \int_0^T f(u(s))\, ds \to \min\,.$$

Existence of minimizers

Let us consider

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Proposition

There exists $\hat{u} \in \mathcal{U}$ such that

$$\mathcal{F}_\Theta(\hat{u}) = \inf_{\mathcal{U}} \mathcal{F}_\Theta.$$

Moreover, for every $\hat{u} \in \arg\min \mathcal{F}_\Theta$, we have $\|\hat{u}\|_{L^p} \leq C(\beta)$.

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In practice, we may be able to construct $\Theta^M \subset \Theta$ such that:

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- $d_H(\Theta^M, \Theta) \to 0$ as $M \to \infty$ (Hausdorff distance).

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Convergence of minimizers. Let $\hat{u}^M \in \arg\min_{\mathcal{U}} \mathcal{F}_{\Theta^M}$.
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Γ-convergence: consequences – improved

Convergence of minima.

$$\min_{u\in\mathcal{U}} \mathcal{F}_{\Theta^M} \to \min_{u\in\mathcal{U}} \mathcal{F}_{\Theta} \quad \text{as } M \to \infty.$$

Convergence of minimizers. Let $\hat{u}^M \in \arg\min_{\mathcal{U}} \mathcal{F}_{\Theta^M}$.
Then, if f is **strictly convex**, $(\hat{u}^M)$ is pre-compact in the **strong topology** of L^p, and clusters are minimizers of $\mathcal{F}_\Theta$.

Convergence of integral costs. Let $\hat{u}^M \in \arg\min_{\mathcal{U}} \mathcal{F}_\Theta$, and assume that $\hat{u}^M \rightharpoonup \hat{u}$. Then,

$$\lim_{M\to\infty} \sup_{\theta\in\Theta^M} a(x^\theta_{\hat{u}^M}(T),\theta) = \sup_{\theta\in\Theta} a(x^\theta_{\hat{u}}(T),\theta),$$

$$\lim_{M\to\infty} \int_0^T f(\hat{u}^M(s))\,ds = \int_0^T f(\hat{u}(s))\,ds.$$

PMP for minimax: notations recap

Notations

For every $u \in \mathcal{U}$, we define $X_u : [0, T] \times \Theta \to \mathbb{R}^n$ as

$$X_u(t, \theta) := x_u^\theta(t),$$

and $\Lambda_u : [0, T] \times \Theta \to (\mathbb{R}^n)^*$ as

$$\Lambda_u(t, \theta) := \lambda_u^\theta(t),$$

where

$$\dot{\lambda}_u^\theta = -\lambda_u^\theta \cdot \frac{\partial}{\partial x} \left(b^\theta(x_u) - A^\theta(x_u) u \right), \quad \lambda_u^\theta(T) = -\nabla_x a(x_u(T), \theta).$$

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Theorem (Vinter, Optimal Control, 2000)

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$$\hat{u}^M(t) \in \arg\max_{v \in \mathbb{R}^m} \left\{ \int_\Theta \Lambda_{\hat{u}^M}(t, \theta) \cdot A^\theta\big(X_{\hat{u}^M}(t, \theta)\big) v \, d\nu^M(\theta) - \beta f(v) \right\}$$

for a.e. $t \in [0, T]$

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for a.e. $t \in [0, T]$, and $\bar{\theta} \in \mathrm{supp}(\nu^M) \implies \bar{\theta} \in \arg\max_{\Theta^M} a(x^\theta_{\hat{u}^M}(T), \theta)$.

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In other words, $\hat{u}^M$ is as well an extremal for

$$\mathcal{G}_{\nu^M}(u) = \int_\Theta a(x^\theta_u(T),\theta)\, d\nu^M(\theta) + \beta \int_0^T f(u(s))\, ds.$$

Γ-convergence and PMP (again)

Consider $\hat{u}^M \in \arg\min \mathcal{F}_{\Theta^M}$, and assume that $\hat{u}^M \to_{L^p} \hat{u} \in \arg\min \mathcal{F}_{\Theta}$.

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$$\begin{cases} \nu^M \rightharpoonup^* \nu & \text{(up to subseq.)} \\ X_{\hat{u}^M} \to_{C^0} X_{\hat{u}} & \\ \Lambda_{\hat{u}^M} \to_{C^0} \Lambda_{\hat{u}} & \\ \hat{u}^M(t) \to \hat{u}(t) & \text{a.e. (up to subseq.)} \end{cases} \qquad \text{as } M \to \infty.$$

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We can pass to the limit here:

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We use that the subdifferential ∂f has closed graph.

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Theorem (S., 2024)

Let $\hat{u} \in \arg\min \mathcal{F}_\Theta$*. Then, considering* $X_{\hat{u}} : [0,T] \times \Theta \to \mathbb{R}^n$ *and* $\Lambda_{\hat{u}} : [0,T] \times \Theta \to (\mathbb{R}^n)^*$ *as before, there exists a probability measure* $\nu \in \mathcal{P}(\Theta)$ *such that*

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for a.e. $t \in [0,T]$, *and* $\bar{\theta} \in \operatorname{supp}(\nu) \implies \bar{\theta} \in \arg\max_{\theta \in \Theta} a(x_{\hat{u}}^\theta(T), \theta)$.

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Reduce to finite ensembles using Γ-convergence, and solve it as a finite-dimensional problem.

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Remark

The minimax problem is harder: the measures μ^M are not explicitly given, they should be *adaptively guessed* during the approximation of the optimal control.

Numerical example

Control system in $\mathbb{R}^2$ affected by uncertainty:

$$\dot{x}^\theta(s) = M^\theta x^\theta(s) + Au(s), \quad x^\theta(0) = 0,$$

where

$$M^\theta := \begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix}, \quad A := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We considered $\theta \in \Theta = [-1/2, 1/2]$, and $\mu \sim \mathrm{Beta}(4, 4)$ centered at 0.

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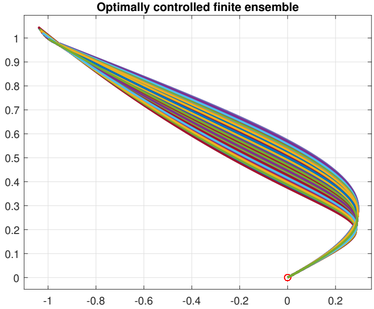
$$\mathcal{G}_\mu(u) := \int_\Theta |x^\theta(1) - y_{\mathrm{tar}}|_2^2 \, d\mu(\theta) + \beta \|u\|_{L^2}^2,$$

$y_{\mathrm{tar}} = (-1, 1)$, with evolution horizon $[0, 1]$.

Numerical example

$N = 300$, $\beta = 10^{-3}$



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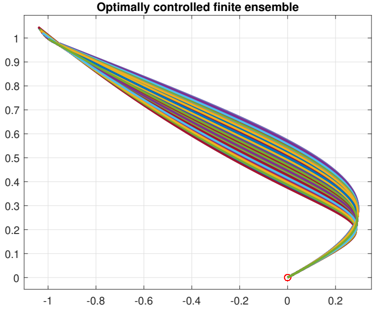

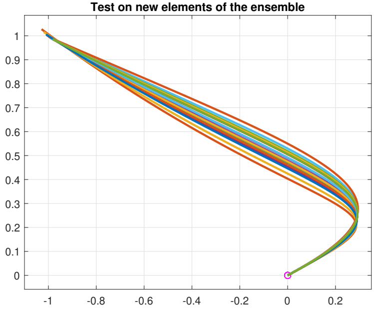

Thanks for the attention!

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