Well-posedness of a non local ocean-atmosphere coupling model

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S. Thery (2023). Well-posedness of a non local ocean-atmosphere coupling model: study of a 1d ekman boundary layer problem with non local kpp-type turbulent viscosity profile

Application of ocean-atmosphere coupling

- Various physical phenomena are governed by the ocean-atmosphere (OA) interaction
- OA models have originally been constructed separately, by two distinct communities.
  $\Rightarrow$ mathematics coherence of such coupling ?

Modelisation

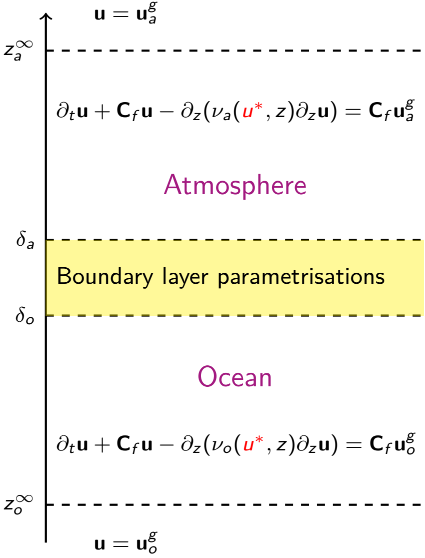
- Translate the realistic OA model in mathematical terms
- Construct a model that take into account account the specificities brought by the numerical models

Work by : Eric Blayo (UGA), Florian Lemarié (Inria-UGA), Charles Pelletier (ECMWF)

Mathematical study of this model

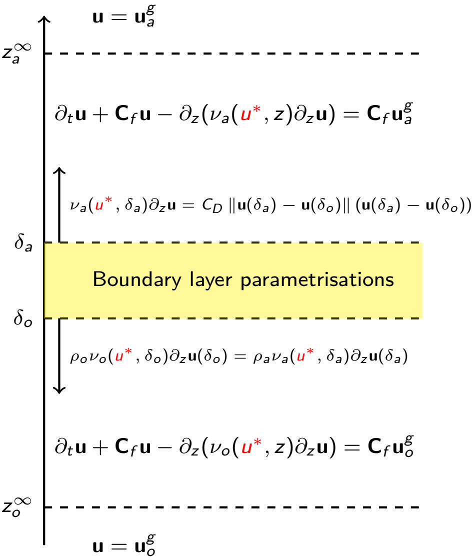
Conditions for the well-posedness (existence and unicity of solutions)

A simplified OA model



- Navier-stokes + **Simplification hypothesys :** (hydrostatic, ect...)
- 1D vertical,
  $\mathbf{u}$ horizontal wind/current
  $\mathbf{C}_f$ Coriolis force
  $\mathbf{u}^g$ source term
- **Boussinesq hypothesys :**
  subgrid-scale parametrisations
  $\Rightarrow \nu$ turbulent viscosity
- **Boundaries layer parametrisation:**
  Interface is a "buffer-zone" where solutions are parameterized

Our local OA model



- Friction law at the interface
- Viscosity parameterized by u^* and z

Our non-local OA model

$$\mathbf{u} = \mathbf{u}_a^g$$

z_a^∞

$$\partial_t \mathbf{u} + \mathbf{C}_f \mathbf{u} - \partial_z(\nu_a(u^*, z)\partial_z \mathbf{u}) = \mathbf{C}_f \mathbf{u}_a^g$$

$$\nu_a(u^*, \delta_a)\partial_z \mathbf{u} = C_D \left\| \mathbf{u}(\delta_a) - \mathbf{u}(\delta_o) \right\| (\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o))$$

δ_a

$$u^* = \sqrt{C_D} \left\| \mathbf{u}(\delta_a) - \mathbf{u}(\delta_o) \right\|$$

δ_o

$$\rho_o \nu_o(u^*, \delta_o)\partial_z \mathbf{u}(\delta_o) = \rho_a \nu_a(u^*, \delta_a)\partial_z \mathbf{u}(\delta_a)$$

$$\partial_t \mathbf{u} + \mathbf{C}_f \mathbf{u} - \partial_z(\nu_o(u^*, z)\partial_z \mathbf{u}) = \mathbf{C}_f \mathbf{u}_o^g$$

z_o^∞

$$\mathbf{u} = \mathbf{u}_o^g$$

- Friction law at the interface
- Viscosity parameterized by u^* and z
- Boundary layer parametrisation: computation of u^* (Pelletier et al. 2021)
- u^* wears the non-locality

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Our goal

- Study the well-posedness of this problem
- Application to OA framework (specific viscosity and order of magnitude)

Existing work and our approch

- Ocean-Atmosphere coupling model : *Lions and al. (1993)*
  - coupling of primitive equations (local problem)
  - not taking into account real numerical scheme in realistic model
- Fluid dynamic community : *Bernardi and al. (2002)*
  - coupling of two turbulent fluids
  - stationary and non local problem
  - very close problem with other kind of viscosity
  - well-posedness depending on the viscosity profile and its variations.

Our approach:

- Use fixed point method on our simplified problem (viscosity parametrized)$\rightarrow$ well-posedness criteria
- Extend the method to non-stationary problem

A fixed point formulation

$$\begin{cases} \partial_t \mathbf{u}_\alpha + \mathbf{C}_f \mathbf{u}_\alpha - \partial_z \left(\nu_\alpha(u^*, z) \partial_z \mathbf{u} \right) & = \mathbf{C}_f \mathbf{u}_\alpha^g \qquad \text{on } (\delta_\alpha, z_\alpha^\infty) \times]0, T[\\ \mathbf{u}_\alpha(z_\alpha^\infty) & = \mathbf{u}_\alpha^g|_{z_\alpha^\infty} \\ \mathbf{u}_\alpha(z, t=0) & = \mathbf{u}_\alpha^0(z) \\ \nu_o \, \partial_z \mathbf{u}_o(\delta_o) & = \lambda^2 \nu_a \, \partial_z \mathbf{u}_a(\delta_a) \\ \nu_a \, \partial_z \mathbf{u}_a(\delta_a) & = C_D \left\| \mathbf{u}(\delta_a) - \mathbf{u}(\delta_o) \right\| \left(\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o) \right) \end{cases}$$

$$u^* = \sqrt{C_D} \left\| \mathbf{u}(\delta_a) - \mathbf{u}(\delta_o) \right\|$$

Strategy for mathematical studying

Write as a fixed point problem :
Show $P : u^* \rightarrow \sqrt{C_D} \left\| \mathbf{u}(\delta_a) - \mathbf{u}(\delta_o) \right\|$ is a contracting mapping.

Our point of view

Sufficient well-posedness criteria on viscosity profile

First step : study the local problem

Suppose u^* is given, the stationary local problem is :

$$\begin{cases} \mathbf{C}_f \mathbf{u}_\alpha - \partial_z \left(\nu_\alpha(u^*, z)\partial_z \mathbf{u}\right) & = \mathbf{C}_f \mathbf{u}_\alpha^g \qquad \text{on } (\delta_\alpha, z_\alpha^\infty) \\ \mathbf{u}_\alpha(z_\alpha^\infty) & = \mathbf{u}_\alpha^g|_{z_\alpha^\infty} \\ \nu_o\, \partial_z \mathbf{u}_o(\delta_o) & = \lambda^2 \nu_a\, \partial_z \mathbf{u}_a(\delta_a) \\ \nu_a\, \partial_z \mathbf{u}_a(\delta_a) & = C_D \left\| \mathbf{u}(\delta_a) - \mathbf{u}(\delta_o) \right\| \left(\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o)\right) \end{cases}$$

Property: Well-posedness of the stationary local problem

Weak formulation of the stationary local problem is well-posed in $H^1(\Omega)$, with $\Omega := \Omega_o \cup \Omega_a$

Proof: Using Galerkine method.
Apriori estimate on $\|\cdot\|_{\mathrm{OA}} \approx \|\cdot\|_{L^2(\Omega_o)} + \|\cdot\|_{L^2(\Omega_a)}$

Fixed point problem

For a given $\mathbf{u}^g$ with "good" regularity, there exist $u^*_{\max}$ such that

$$P : \begin{cases} [0, u^*_{\max}] & \rightarrow & [0, u^*_{\max}] \\ u^* & \rightarrow & \sqrt{C_D}\,\|\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o)\| \\ & & \mathbf{u} \text{ sol. local pb with } \nu(u^*) \end{cases}$$

Sufficient well-posedness criteria

P is contractant if $\forall u^*, v^* \in [0, u^*_{\max}]$:

$$\max_{z\in\Omega_\alpha} \left| \frac{\nu_\alpha(z,u^*) - \nu_\alpha(z,v^*)}{\sqrt{\nu_\alpha(z,u^*)\nu_\alpha(z,v^*)}} \right| \leq (\mathcal{M}^e)^{-1} \left|u^* - v^*\right|^{3/2}$$

with $\mathcal{M}^e$ an upper bound of $\|\sqrt{\nu}\partial_z \mathbf{u}\|_{\mathrm{OA}}$ for all $\mathbf{u}$ solution of the local problem.

Fixed point problem

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with $\mathcal{M}^e$ an upper bound of $\|\sqrt{\nu}\partial_z \mathbf{u}\|_{\mathrm{OA}}$ for all $\mathbf{u}$ solution of the local problem.
Another more restrictive condition can be derived as

$$\|\partial_{u^*}\nu_\alpha\|_{\mathcal{L}^\infty(\Omega,[0,u^*_{\max}])} \leq C \min(\nu_\alpha) \left(\mathcal{M}^e\right)^{-1} \qquad C \text{ cst obtain via trace thm}$$

Same criteria for the non stationary problem

For a given $\mathbf{u}^g$ with "good" regularity, there exist $u^*_{\max}$ such that

$$P : \begin{cases} \mathcal{V}^* & \rightarrow & \mathcal{V}^* \\ u^*(t) & \rightarrow & \sqrt{C_D}\,\|\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o)\|\,(t) \\ & & \mathbf{u} \text{ sol. local pb with } \nu(u^*) \end{cases}$$

with $\mathcal{V}^* := \left\{ u^* \in \mathcal{C}^1([0,T]), 0 \leq u^*(t) \leq u^*_{\max} \right\}$

Property: Well-posedness criteria

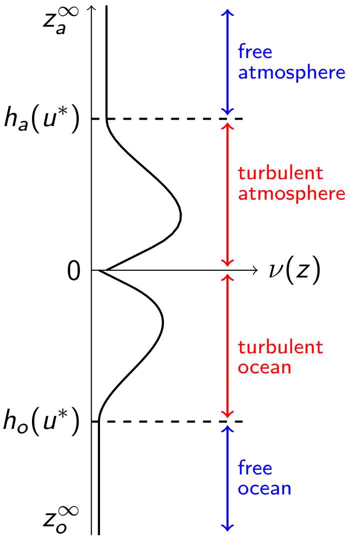
P is contractant if $\forall u^*, v* \in [0, u^*_{\max}]$:

$$\max_{z\in\Omega_\alpha} \left| \frac{\nu_\alpha(z,u^*) - \nu_\alpha(z,v^*)}{\sqrt{\nu_\alpha(z,u^*)\nu_\alpha(z,v^*)}} \right| \leq (\mathcal{M})^{-1} \left|u^* - v^*\right|^{3/2}$$

with $\mathcal{M}$ an upper bound of $\sup_{t\in[0,T]} \|\sqrt{\nu}\partial_z \mathbf{u}\|_{OA}$ for all $\mathbf{u}$ solution of the local problem.

Justification of the choice of space $\mathcal{V}^*$: $\mathcal{M}$ exist if $\|\partial_t \nu\| \in \mathcal{L}^\infty(\Omega\times]0,T[)$

KPP viscosity



- Turbulent part : influenced by u^*
  Free part : $\nu_\alpha = \nu_\alpha^m$
- Turbulence layer thickness : $h_\alpha(u^*) = c_\alpha u^*$
- Coherence with interface boundary parametrisation : $\nu(\delta_\alpha) \approx \kappa u^* |\delta_\alpha|$

Example *O'Brien, (1970)*

$$\nu_\alpha(u^*, z) = \kappa u^* |z| \left(1 - \frac{z}{h_\alpha}\right)^2 H\left(1 - \frac{z}{h_\alpha}\right) + \nu_\alpha^m$$

with H heaviside function

Application to OA order of magnitude

Example with stationary state and $\mathbf{u}_\alpha^g$ constant : $\mathcal{M}^e = \frac{C_D}{2} \left\|\mathbf{u}_a^g - \mathbf{u}_o^g\right\|^{3/2}$

- Criteria of well-posedness :

$$\left\|\partial_{u^*} \nu_\alpha\right\|_{\mathcal{L}^\infty(\Omega,[0,u^*_{\max}])} \leq 2.5 \times 10^{-5} \left\|\mathbf{u}_a^g - \mathbf{u}_o^g\right\|^{3/2}$$

- OA order of magnitude $\|\mathbf{u}_a^g - \mathbf{u}_o^g\| \approx 10\mathrm{ms}^{-1}$ give very small upper bound for $\|\partial_{u^*} \nu_\alpha\|$
- KPP O'Brien viscosity : $h_\alpha(u^*) \leq 2.5 \times 10^{-5} \left\|\mathbf{u}_a^g - \mathbf{u}_o^g\right\|^{3/2}$ that is possible if $\|\mathbf{u}_a^g - \mathbf{u}_o^g\| \leq 10^{-4}\mathrm{ms}^{-1}$

Cause of non unicity ?

- Due to the too large bounding in the criterion?
- Due to the profile of ν and OA order of magnitude ?

A necessary and sufficient criteria for stationary problem

Property

We we can solve the ODE :

$$\begin{aligned}
\mathbf{C}_f \mathbf{u}_\alpha - \partial_z \left(\nu_\alpha(u^*, z)\partial_z \mathbf{u}\right) &= \mathbf{C}_f \mathbf{u}_\alpha^g \qquad \text{on } (\delta_\alpha, z_\alpha^\infty) \\
\mathbf{u}_\alpha(z_\alpha^\infty) &= \mathbf{u}_\alpha^g|_{z_\alpha^\infty} \\
\nu_o\, \partial_z \mathbf{u}_o(\delta_o) &= \lambda^2 \nu_a\, \partial_z \mathbf{u}_a(\delta_a) \\
\nu_a\, \partial_z \mathbf{u}_a(\delta_a) &= C_D \left\|\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o)\right\| \left(\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o)\right) \\
u^* &= \sqrt{C_D} \left\|\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o)\right\|
\end{aligned}$$

We can write

$$|\mathbf{u}_a^g - \mathbf{u}_o^g\| = F(u^*)$$

.

Sufficient and necessary well-posedness criteria

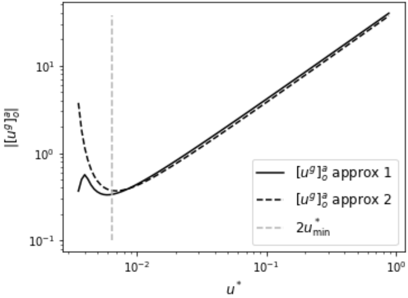
If F in continuous and injective on I^* then non-local problem is well-posed on $u^* \in I^*$

No unicity of solution in OA order of magnitude

Exemple

KPP viscosity profile approximate by $P^2(z)$ polynomial.

- $I^* := [u^*_{\min}, u^*_{\max}]$ depending of z^∞_α and δ_α.
- F is combination of Legendre polynomial (complicated formula)
- OA order of magnitude : we can prove that we have a inflexion point when $u^* \approx 2u^*_{\min}$



Conclusion

- **Wellposedness criteria** :
  - Problem re-write as a fixed point formulation
  - Sufficient well-posedness condition depending on viscosity profile and its variation
  - Variation $\|\partial_{u^*}\nu\|$ small compare to $\min(\nu)$
- **Non unicity of solution in OA frame work**
  - Sufficient well-posedness criteria non verify in the OA order of magnitude
  - Necessary and sufficient condition for stationary state with KPP viscosities
    $\rightarrow$ non unicity of solution for small value of u^*
- **Improvement**
  - Use alternative boundary conditions
  - More realistic boundary layer parametrisation $\rightarrow$ C_D depending on u^*
  - Find another method than the fixed point formulation