Numerical analysis of a spectral problem equation with Ventcel boundary conditions on curved meshes

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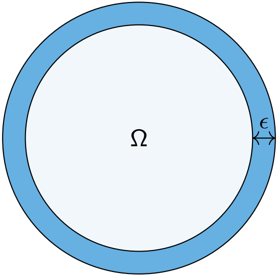
Section 1

Motivation

Vibrating mechanical piece

$\Longrightarrow$ **Shape optimisation**

Motivation



[1] V. Bonnaillie-Noel, D. Brancherie, M. Dambrine, F. Herau, S. Tordeux, and G. Vial, *Multiscale expansion and numerical approximation for surface defects.* ESAIM (2011).

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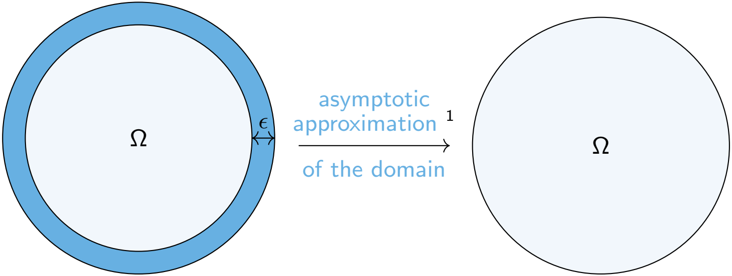

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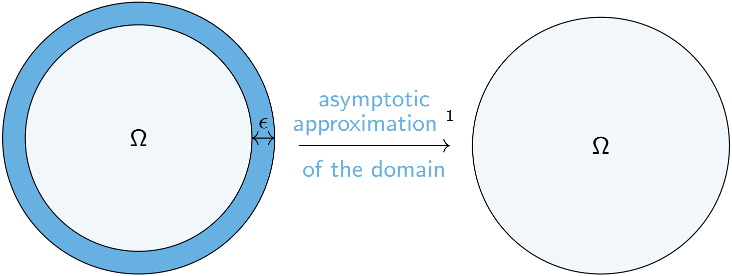

Ventcel boundary condition
(G.I.B.C, generalisation of Robin,
higher order).

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Definitions

We can define the **Laplace-Beltrami operator** of $w \in C^2(\Gamma)$ by

$$\Delta_\Gamma w = \mathrm{div}_\Gamma[\nabla_\Gamma w],$$

where

1. div_Γ is the tangential divergence.
2. $\nabla_\Gamma w$ is the tangential gradient of w given by

$$\nabla_\Gamma w = \nabla w^\ell - (\nabla w^\ell \cdot \mathbf{n})\mathbf{n},$$

   where w^ℓ is a smooth extension of w on $\mathbb{R}^n$ and $\mathbf{n}$ is a unit normal on Γ.

Section 2

The spectral Ventcel problem

Let:

- Ω be an open-bounded connected domain of $\mathbb{R}^n$ (n=2,3);
- $\Gamma = \partial\Omega$ be a smooth boundary.

We consider the following system,

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ -\Delta_\Gamma u + \partial_n u + u = 0 & \text{on } \Gamma, \end{cases}$$

where

- $\mathbf{n}$ is the outer unit normal vector on Γ;
- Δ_Γ Laplace-Beltrami operator.

Variational form

The variational form of the problem is: to find $(\lambda, u) \in \mathbb{R} \times \mathrm{H}^1(\Omega, \Gamma)$ such that,

$$a(u, v) = \lambda \int_\Omega uv dx,$$

for all $v \in \mathrm{H}^1(\Omega, \Gamma) := \{u \in \mathrm{H}^1(\Omega),\ u_{|_\Gamma} \in \mathrm{H}^1(\Gamma)\}$, where the bilinear form a is given by,

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx + \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v d\sigma + \int_\Gamma uv d\sigma.$$

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Remark

There exists an increasing sequence $(\lambda_n)_{n\geq 1}$ of positive eigenvalues, tending to infinite with finite multiplicities. Their associated eigenfunctions form an orthonormal Hilbert basis of $\mathrm{L}^2(\Omega)$, denoted $(u_n)_{n\geq 1} \in \mathrm{H}^1(\Omega, \Gamma)$.

Section 3

Motivation behind curved higher order meshes

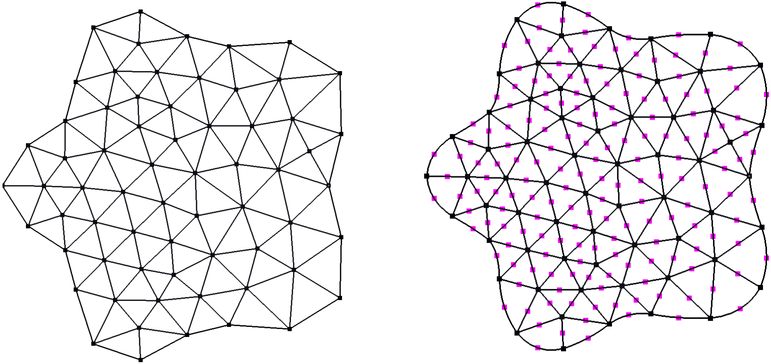


Figure: Affine and quadratic meshes of a smooth domain **with the same number of mesh elements**.

- Not exact meshes !
- Error decrease on higher order meshes.

The discrete variational form

We define the finite element space,

$$\mathbb{V}_h = \{\chi \in C^0(\Omega_h),\ \chi|_T = \hat{\chi} \circ (F_T^{(r)})^{-1};\ \hat{\chi} \in \mathbb{P}^k(\hat{T}),\ \forall\ T \in \mathcal{T}_h^{(1)}\}.$$

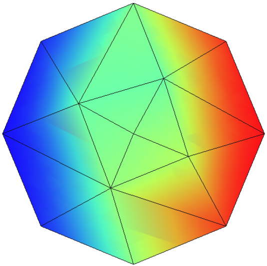
The approximation problem: find $(\Lambda, U) \in \mathbb{R} \times \mathbb{V}_h$ such that,

$$a_h(U, V) = \Lambda \int_{\Omega_h} UV \mathrm{d}x, \qquad \forall\ V \in \mathbb{V}_h,$$

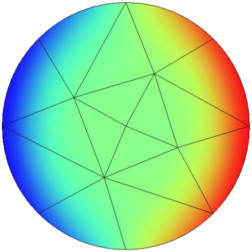
where a_h is the following bilinear form, defined on $\mathbb{V}_h \times \mathbb{V}_h$,

$$a_h(U, V) := \int_{\Omega_h} \nabla U \cdot \nabla V \mathrm{d}x + \int_{\Gamma_h} \nabla_{\Gamma_h} U \cdot \nabla_{\Gamma_h} V \mathrm{d}\sigma_h + \int_{\Gamma_h} UV \mathrm{d}\sigma_h.$$

Lift motivation



$$\xrightarrow{G_h^{(r)}}$$



$$\boxed{u_h : \Omega_h \to \mathbb{R} \qquad \Longrightarrow \qquad u_h^\ell : \Omega \to \mathbb{R}.}$$

Advantages:

- redefining a function on another domain;
- intervenes in the error estimation.

Lift references throughout the years

The idea of lifting a function from the discrete domain onto the continuous one,

- is not new;
- dates back to the 1970's;
- most notably: Nedelec[2], Scott[3], Lenoir[4], Bernardi[5].

[2] J.-C. Nedelec, *Curved finite element methods for the solution of singular integral equations on surfaces in R3,* CMAME (1976).

[3] L. R. Scott, *Interpolated boundary conditions in the finite element method*, SINUM (1975).

[4] M. Lenoir, *Optimal isoparametric finite elements and error estimates for domains involving curved boundaries,* SINUM (1986).

[5] C. Bernardi, *Optimal finite-element interpolation on curved domains,* SINUM (1989).

[6] G. Dziuk, *Finite elements for the Beltrami operator on arbitrary surfaces, in Partial differential equations and calculus of variations,* (1988).

[7] A. Demlow, *Higher-order finite element methods and pointwise error estimates for elliptic646 problems on surfaces,* SINUM (2009)

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Surface lifts;

- were firstly introduced in 1988 by Dziuk [6];
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Recently, **Elliott and Ranner** discussed a simpler way to define a lift, in their work: *Finite element analysis for a coupled bulk-surface partial differential equation,* IMA J. Numer. Anal. (2013).

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[2] J.-C. Nedelec, *Curved finite element methods for the solution of singular integral equations on surfaces in R3,* CMAME (1976).

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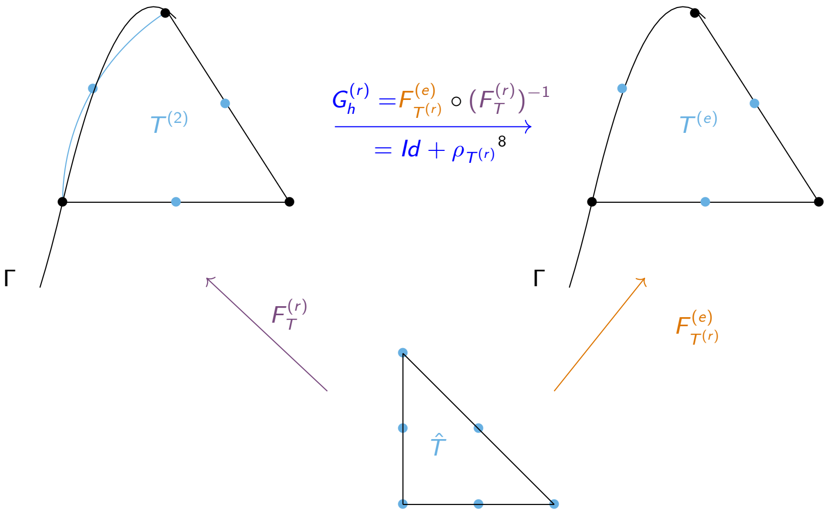
Modified Lift transformation

For $r = 2$:



[8] F. Caubet, J. Ghantous, C. Pierre, *A priori error estimates of a diffusion equation with Ventcel boundary conditions on curved meshes,* (accepted in SIAM J. Num. Anal.).

Lift operator

Definition

Let $u_h \in \mathrm{L}^2(\Omega_h, \Gamma_h)$. We define the lift associated to u_h, denoted $u_h^\ell \in \mathrm{L}^2(\Omega, \Gamma)$, by,

$$\boxed{u_h^\ell \circ G_h^{(r)} = u_h.}$$

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Remark

The restriction to $\Gamma_h = \partial\Omega_h$ *of* **the new transformation** *is equal to the orthogonal projection* b*:*

$$G_h^{(r)}{}_{|_{\Gamma_h}} = b : \Gamma_h \to \Gamma.$$

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Remark

Note that,

$$\boxed{Tr(\, u_h^\ell \,) = (\, Tr(u_h)\,)^\ell.}$$

Lift $\rightsquigarrow$ Error estimation (Main result)

Section 6

Eigenvalue error[9]

Theorem

Let

- λ *be an exact eigenvalue of multiplicity* N *such that,*

$$\lambda = \lambda_j \quad \forall j \in \{i, ..., i+N-1\};$$

- Λ_j *be a discrete eigenvalue.*

Then, for any $j \in \{i, ..., i+N-1\}$*, there exists* $c_\lambda > 0$ *such that,*

$$\boxed{|\lambda_j - \Lambda_j| \leq c_\lambda(h^{2k} + h^{r+1}),}$$

where

- h *is the mesh size;*
- k *is the degree of the Lagrangien Finite element space;*
- r *is the geometrical order of the mesh.*

[9] F. Caubet, J. Ghantous, C. Pierre, *Finite element analysis of a spectral problem on curved meshes occurring in diffusion with high order boundary conditions* (submitted).

Eigenfunctions errors[10]

Theorem

Let

- $\{u_j\}_{j=i}^{i+N-1}$ *be the exact eigenfunctions associated to* λ*;*
- E^ℓ *be the* **lifted** *discrete eigenfunction space associated to* Λ_j *that approximate* λ*;*
- U *be the orthogonal projection of* u_j *over* E^ℓ*.*

Then, for any $j \in \{i, ..., i+N-1\}$*, there exists* $c_\lambda > 0$ *such that,*

$$\inf_{U \in E^\ell} \|u_j - U\|_{\mathrm{L}^2(\Omega)} \leq c_\lambda (h^{k+1} + h^{r+1/2}), \qquad \inf_{U \in E^\ell} \|u_j - U\|_{\mathrm{H}^1(\Omega,\Gamma)} \leq c_\lambda (h^{k} + h^{r+1/2}),$$

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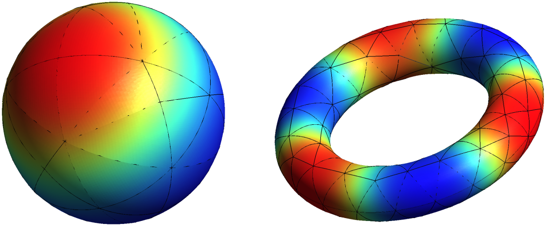
Section 7

CUMIN

Curved Meshes In Numerical simulations [11] is a finite element library including:

- finite elements of high order,
- curved geometries: meshes of high geometrical order,
- solvers for diffusion and linear elasticity PDEs,
- coupling with linear system solvers and eigenvalue problem solvers.



[11] https://plmlab.math.cnrs.fr/cpierre1/cumin.

Eigenvalue estimations on a smooth domain

Consider the following system,

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ -\Delta_\Gamma u + \partial_n u + u = \lambda u & \text{on } \Gamma. \end{cases}$$

We do not know the exact eigenvalues of this system on the flower domain. A convergence analysis is performed on the 6^{th} eigenvalue λ_6 computed on a reference cubic mesh.

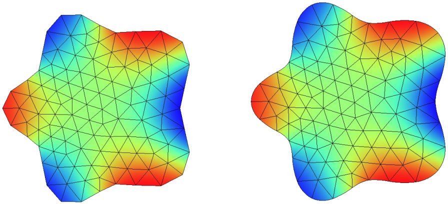


Figure: Numerical eigenfunction U_6 on affine and quadratic meshes.

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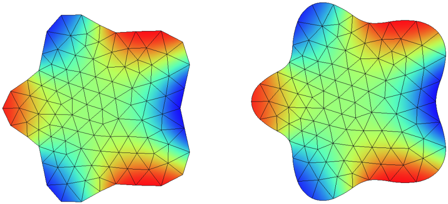


Figure: Numerical eigenfunction U_6 on affine and quadratic meshes.

The reference mesh of order $r = 3$ using a $\mathbb{P}^4$ FEM counts 20×2^5 boundary edges and is made of approximately 77 000 cubic triangles, the associated $\mathbb{P}^4$ finite element space has approximately 610 000 DOF.

Eigenvalue estimations

A convergence analysis is performed on the 6^{th} eigenvalue λ_6.
The error $|\lambda_6 - \Lambda_6|$ computed are computed on a series of succesively refined meshes: each mesh counts $20 \times 2^{n-1}$ edges on the domain boundary, for $n = 1 \dots 5$.

	$\|\lambda_6 - \Lambda_6\|$			
Mesh type	$\mathbb{P}^1$	$\mathbb{P}^2$	$\mathbb{P}^3$	$\mathbb{P}^4$
Affine (r=1)	1.99	2.004	2.003	2.001
Quadratic (r=2)	2.002	4.003	4.07	3.99
Cubic (r=3)	2.00	3.05	4.07	3.99

Table: Convergence order of $|\lambda_6 - \Lambda_6|$

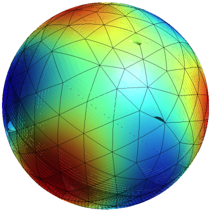
The convergence order of $|\lambda_6 - \Lambda_6|$ is equal to $\boxed{\min\{2k, r+1\}}$.

Numerical results in 3d on quadratic meshes

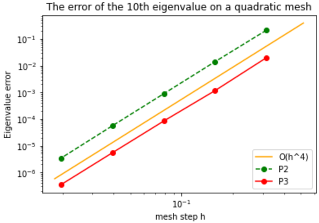
Consider the following system on the unit ball;

$$\begin{cases} -\Delta u = & 0 \quad \text{in } \mathrm{B}(0,1), \\ -\beta\Delta_\Gamma u + \partial_{\mathrm{n}} u = & \lambda u \quad \text{on } \mathrm{S}(0,1). \end{cases}$$

A convergence analysis is performed on the 10^{th} eigenvalue λ_{10} of multiplicity 7 with corresponding eigenspace E_3, which is equal to the space of harmonic polynomials of degree 3.

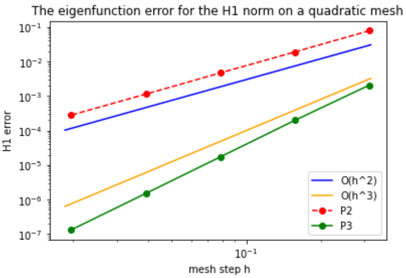


A quadratic mesh (r=2) of the unit Ball.



The convergence rate of $e_{\lambda_{10}} = |\lambda_{10} - \Lambda_{10}| = \min\{2k, r+1\}$.

Numerical results in 3d on quadratic meshes



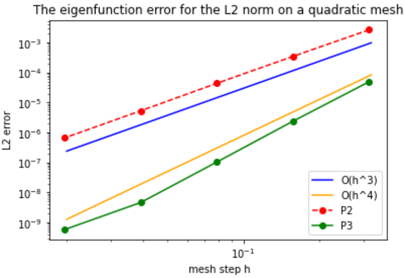


Figure: Display of the convergence rate of $e_{\mathrm{H}_0^1}$ (resp. e_{L^2}) using $\mathbb{P}^2$ and $\mathbb{P}^3$ finite element on an quadratic mesh on the left (resp. right).

$$\text{CV rate of } e_{\mathrm{H}_0^1} = \inf\{\|\nabla(U_{10}^{\ell} - u)\|_{\mathrm{L}^2(\Omega)}, u \in \mathrm{E}_3\} = \min\{k, r + 1/2\}.$$

$$\text{CV rate of } e_{\mathrm{L}^2} = \inf\{\|U_{10}^{\ell} - u\|_{\mathrm{L}^2(\Omega)}, u \in \mathrm{E}_3\} = \min\{k+1, r + 1/2\} <= \min\{k+1, r+1\}.$$

Section 8

Perspectives

Ongoing work:

- Do an a priori error estimation of the **Elasticity** model theoretically and numerically,

$$\begin{cases} -\mathrm{div}(Ae(u)) = f & \text{in } \Omega, \\ -\mathrm{div}_\Gamma(A_\Gamma e_\Gamma(u)) + u + Ae(u)\mathbf{n} = g & \text{on } \Gamma, \end{cases}$$

- Optimisation of the eigenvalues of the spectral problem under Volume Constraint.

Long term:

- Optimisation of the Eigenvalues of an **Elasticity** model under Volume constraint,

$$\begin{cases} -\mathrm{div}(Ae(u)) = \lambda u & \text{in } \Omega, \\ -\mathrm{div}_\Gamma(A_\Gamma e_\Gamma(u)) + u + Ae(u)\mathbf{n} = 0 & \text{on } \Gamma, \end{cases}$$

Thank you for your attention