About the dynamics of the Landau-Lifshitz equation

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Introduction

The Landau-Lifshitz equation

$$\partial_t m + m \times \Big(\Delta m - J(m)\Big) = 0,$$

describes an Hamiltonian dynamics for the magnetization

$$m = (m_1, m_2, m_3) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{S}^2,$$

in a ferromagnetic material (Landau and Lifshitz [35]). In this equation, the diagonal matrix

$$J = \begin{pmatrix} j+\lambda_1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j+\lambda_3 \end{pmatrix},$$

gives account of the anisotropy of the material.

In dimension $N = 1$, the Landau-Lifshitz equation is integrable by means of the inverse scattering method (Sklyanin [79]) :

- this equation owns a form of universality (see e.g. Faddeev and Takhtajan [87]) in the sense that it contains other integrable equations in suitable asymptotic regimes.

- its long-time dynamics is governed by the propagation of a train of solitons plus a dispersive part.

I. Asymptotic models in higher dimensions

Deriving well-known nonlinear dispersive models from the Landau-Lifshitz equation remains possible in any higher dimensions. A typical example is given by the Sine-Gordon regime.

This regime describes the dynamics of a biaxial material in an uniaxial regime with strong planar anisotropy. The numbers λ_1 and λ_3 are given by

$$\lambda_1 = \sigma\varepsilon \quad \text{and} \quad \lambda_3 = \frac{1}{\varepsilon},$$

where $\varepsilon > 0$ is a small parameter and $\sigma > 0$ is a fixed constant.

Its derivation relies on an hydrodynamical formulation. When the map $\check{m} = m_1 + i m_2$ does not vanish, it may be lifted as

$$\check{m} = \sqrt{1 - m_3^2}\Big(\sin(\phi) + i\cos(\phi)\Big).$$

The hydrodynamical variables

$$(\phi, u = m_3),$$

satisfy the hydrodynamical Landau-Lifshitz equation

$$\begin{cases}\partial_t \phi = -\mathrm{div}\Big(\frac{\nabla u}{1-u^2}\Big) + \frac{u|\nabla u|^2}{(1-u^2)^2} - u|\nabla \phi|^2 + u\Big(\lambda_3 - \lambda_1 \sin^2(\phi)\Big),\\ \partial_t u = \mathrm{div}((1 - u^2)\nabla \phi) - \frac{\lambda_1}{2}(1 - u^2)\sin(2\phi).\end{cases}$$

The long-wave change of variables

$$\Phi_\varepsilon(x,t) = \phi\Big(\frac{x}{\sqrt{\varepsilon}}, t\Big) \text{ and } U_\varepsilon(x,t) = \frac{1}{\varepsilon} u\Big(\frac{x}{\sqrt{\varepsilon}}, t\Big),$$

leads to the system

$$\begin{cases} \partial_t \Phi_\varepsilon = U_\varepsilon\Big(1 - \varepsilon^2\Big(\sigma \sin^2(\Phi_\varepsilon) + |\nabla \Phi_\varepsilon|^2\Big)\Big) \\ \qquad\qquad -\mathrm{div}\Big(\frac{\varepsilon^2 \nabla U_\varepsilon}{1-\varepsilon^2 U_\varepsilon^2}\Big) + \frac{\varepsilon^4 U_\varepsilon |\nabla U_\varepsilon|^2}{(1-\varepsilon^2 U_\varepsilon^2)^2}, \\ \partial_t U_\varepsilon = \mathrm{div}\Big((1 - \varepsilon^2 U_\varepsilon^2)\nabla \Phi_\varepsilon\Big) - \frac{\sigma}{2}(1 - \varepsilon^2 U_\varepsilon^2) \sin(2\Phi_\varepsilon). \end{cases} \tag{HLL$_\varepsilon$}$$

When $\varepsilon \to 0$, the limit function Φ satisfies the Sine-Gordon equation

$$\partial_{tt}\Phi - \Delta\Phi + \frac{\sigma}{2}\sin(2\Phi) = 0. \tag{SG}$$

Theorem (de Laire and G. [17]). *Let $k > N/2$ and $0 < \varepsilon < 1$. Consider an initial datum (Φ_0, U_0) such that the quantity*

$$\kappa_0 = \|\nabla\Phi_0\|_{H^{k+3}} + \varepsilon\|\nabla U_0\|_{H^{k+3}} + \|\sin(\Phi_0)\|_{H^{k+3}} + \|U_0\|_{H^{k+3}},$$

satisfies the condition $C\,\varepsilon\,\kappa_0 \leq 1$ for a positive number C. There exists a number

$$T_\varepsilon \geq \frac{1}{C\kappa_0^2},$$

such that the corresponding solutions $(\Phi_\varepsilon, U_\varepsilon)$ to (HLL_ε), *and (Φ, U) to* (SG), *satisfy*

$$\|\sin(\Phi_\varepsilon(\cdot,t) - \Phi(\cdot,t))\|_{L^2} + \|\nabla\Phi_\varepsilon(\cdot,t) - \nabla\Phi(\cdot,t)\|_{H^{k-1}} + \|U_\varepsilon(\cdot,t) - U(\cdot,t)\|_{H^k} \leq C\,\varepsilon^2\,\kappa_0(1+\kappa_0)^3\,e^{C(1+\kappa_0)^2\,t},$$

for any $0 \leq t \leq T_\varepsilon$.

(see also Shatah and Zeng [06], Chiron [14], Germain and Rousset [16], and de Laire and G. [21])

II. Stability of solitons for the Landau-Lifshitz equation with an easy-plane anisotropy

1. Link with the Gross-Pitaevskii equation

For $\lambda_1 = 0$ and $\lambda_3 = 1$, the Landau-Lifshitz equation with an easy-plane anisotropy writes as

$$\partial_t m + m \times \Big(\Delta m - m_3 e_3 \Big) = 0,$$

with $e_3 = (0, 0, 1)$.

When the map $\check{m} := m_1 + i m_2$ is lifted as $\check{m} = \sqrt{1 - m_3^2}\, e^{i\varphi}$, the variables $v := m_3$ and $w := \nabla\varphi$ solve the hydrodynamical system

$$\begin{cases} \partial_t v = -\mathrm{div}\Big((1 - v^2) w\Big), \\ \partial_t w = -\nabla\Big(v - v|w|^2 - \dfrac{\Delta v}{1 - v^2} - \dfrac{v|\nabla v|^2}{(1 - v^2)^2}\Big). \end{cases} \tag{HLL}$$

This system is very similar to the one corresponding to the Gross-Pitaevskii equation

$$i\partial_t \Psi + \Delta \Psi + \Psi(1 - |\Psi|^2) = 0,$$

given for a function $\Psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$.

When this function is indeed lifted as $\Psi := \sqrt{\rho}\, e^{i\varphi}$, the variables $\eta := 1 - \rho$ and $v := -\nabla\varphi$ solve the hydrodynamical Gross-Pitaevskii equation

$$\begin{cases} \partial_t \eta = -2\text{div}\Big((1-\eta)v\Big), \\ \partial_t v = -\nabla\Big(\eta - |v|^2 - \frac{\Delta\eta}{2(1-\eta)} - \frac{|\nabla\eta|^2}{4(1-\eta)^2}\Big). \end{cases}$$

2. Travelling-wave solutions

Travelling waves are special solutions of the form

$$m(x,t) = m_c(x_1 - c\,t, \dots, x_N).$$

Their profile m_c is solution to the nonlinear elliptic equation

$$\Delta m_c + \Big(|\nabla m_c|^2 + [m_c]_3^2\Big) m_c - [m_c]_3\, e_3 + c\, m_c \times \partial_1 m_c = 0.$$

In dimension $N = 1$, (non-constant) travelling waves are called dark solitons. For any speed $|c| < 1$, there exists a unique soliton (up to the geometric invariances), whose expression is explicit.

When $c \neq 0$, the soliton m_c can be identified in the hydrodynamical formulation with the pair

$$\mathfrak{v}_c(x) := \left(v_c(x) = \frac{\sqrt{1 - c^2}}{\cosh\left(\sqrt{1 - c^2} x\right)}, w_c(x) = \frac{c\, v_c(x)}{1 - v_c(x)^2} \right).$$

A train of solitons is then defined as a perturbation of a sum of solitons

$$S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}(x) := \sum_{j=1}^{N} s_j\, \mathfrak{v}_{c_j}(x - a_j),$$

for parameters $\mathfrak{a} \in \mathbb{R}^N$, $\mathfrak{c} \in (-1, 1)^N$ and $\mathfrak{s} \in \{\pm 1\}^N$.

3. Asymptotic stability of well-prepared trains of solitons

Theorem (Bahri [18]). *Let* $\mathfrak{s}_0 \in \{\pm 1\}^N$ *and* $\mathfrak{c}_0 \in (-1, 1)^N$ *with*

$$[c_0]_1 < \cdots < 0 < \cdots < [c_0]_N.$$

There exist two numbers $\alpha_* > 0$ *and* $L_* > 0$*, such that, if an initial datum* $\mathfrak{v}_0 = (v_0, w_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ *satisfies*

$$\Big\| \mathfrak{v}_0 - S_{\mathfrak{c}_0, \mathfrak{a}_0, \mathfrak{s}_0} \Big\|_{H^1 \times L^2} = \alpha_0 < \alpha_*,$$

for positions $\mathfrak{a}_0 \in \mathbb{R}^N$ *such that*

$$\min_{1 \leq k \leq N-1} \Big([a_0]_{k+1} - [a_0]_k \Big) = L_0 > L_*,$$

then there exist positions $a_k \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$, *with* $a_k(0) = [a_0]_k$, *and speeds* $\sigma_k \in (-1, 1)$, *with* $\sigma_k \neq 0$, *and,*

$$a_k'(t) \underset{t \to +\infty}{\to} \sigma_k,$$

such that the unique solution $\mathfrak{v}$ *of* (HLL) *with initial datum* $\mathfrak{v}_0$ *satisfies*

$$\mathfrak{v}(\cdot + a_k(t), t) \underset{t \to +\infty}{\rightharpoonup} [s_0]_k \, \mathfrak{v}_{\sigma_k} \quad \text{in } H^1(\mathbb{R}) \times L^2(\mathbb{R}),$$

for all $1 \leq k \leq N$.

(See also Martel, Merle and Tsai [02], Bethuel, G. and Smets [14, 14], G. and Smets [15], de Laire and G. [15], Bahri [16]).

4. In higher dimensions

In dimensions $N = 2$ and $N = 3$, Papanicolaou and Spathis [99] numerically exhibited a branch of travelling waves with speeds $|c| < 1$.

Lin and Wei [10] proved the existence of small speed travelling waves in dimension $N = 2$.

There are much more theoretical results concerning the existence and orbital stability of branches of travelling waves for the Gross-Pitaevskii equation.

(See e.g. Jones, Putterman and Roberts [82, 86], Bethuel, G. and Saut [09], Maris [13], Chiron and Pacherie [21, 23, 23], ...).

In order to improve the dynamical description of the Landau-Lifshitz equation, it is interesting to investigate numerically :

- the existence of multiple branches of travelling waves in $\mathbb{R}^2$ or in $\mathbb{R}^3$ (in the spirit of Chiron and Scheid [16, 18]),

- the dynamical interactions between travelling waves in $\mathbb{R}^2$ or in $\mathbb{R}^3$ (in the spirit of the description of the vortex motion for the Gross-Pitaevskii equation).

Thank you very much !