Arbitrary high order Discontinuous Galerkin schemes: variants and application to advection-diffusion equations

▶ Interior Penalty vs. Cattaneo relaxation methods ◀

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Advection-diffusion equations

2D unsteady nonlinear **advection-diffusion** on $[x^-, x^+] \times [y^-, y^+] \times [0, T_f]$

$$\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u} - \nu \nabla \mathbf{u}) = \mathbf{S}, \tag{1}$$

with $\mathbf{u} = [u, v]^T$ and $\mathbf{S} = [S_u, S_v]^T$. If we introduce the notations

$$\mathbf{F}_a = u\mathbf{u}, \qquad \mathbf{F}_d = -\nu \partial_x \mathbf{u}, \qquad \mathbf{F} = \mathbf{F}_a + \mathbf{F}_d = [F_u, F_v]^T, \tag{2}$$

$$\mathbf{G}_a = v\mathbf{u}, \qquad \mathbf{G}_d = -\nu \partial_y \mathbf{u}, \qquad \mathbf{G} = \mathbf{G}_a + \mathbf{G}_d = [G_u, G_v]^T, \tag{3}$$

equation (1) can be written

$$\partial_t \mathbf{u} + \partial_x \mathbf{F} + \partial_y \mathbf{G} = \mathbf{S}. \tag{4}$$

Discontinuous Galerkin schemes

DG variational formulation for the 1D conservation law $\partial_t \mathbf{u} + \partial_x \mathbf{F}(\mathbf{u}) = \mathbf{S}$

$$
\begin{aligned}
\int_{\omega_i} \theta_k(x)\mathbf{u}^{n+1}(x)dx = & \int_{\omega_i} \theta_k(x)\mathbf{u}^{n}(x)dx + \int_{\omega_i}\int_{t^n}^{t^{n+1}} \partial_x\theta_k(x)\mathbf{F}(\mathbf{u})(x,t)dxdt \\
& - \left(\int_{t^n}^{t^{n+1}} \theta_k(x_{i+1})\tilde{\mathbf{F}}(\tilde{\mathbf{u}})(x_{i+1},t)dt - \int_{t^n}^{t^{n+1}} \theta_k(x_i)\tilde{\mathbf{F}}(\tilde{\mathbf{u}})(x_i,t)dt \right) \\
& + \varepsilon \left(\int_{t^n}^{t^{n+1}} F_d(\theta_k)(x_{i+1})\tilde{\mathbf{u}}(x_{i+1},t)dt - \int_{t^n}^{t^{n+1}} F_d(\theta_k)(x_i)\tilde{\mathbf{u}}(x_i,t)dt \right) \\
& + \int_{\omega_i}\int_{t^n}^{t^{n+1}} \theta_k(x)\mathbf{S}(x,t)dxdt. \qquad (5)
\end{aligned}
$$

Discontinuous Galerkin schemes

What about high order in time and numerical fluxes?

$$\begin{aligned}\int_{\omega_i} \theta_k(x)\mathbf{u}^{n+1}(x)dx = &\int_{\omega_i} \theta_k(x)\mathbf{u}^n(x)dx + \int_{\omega_i}\int_{t^n}^{t^{n+1}} \partial_x\theta_k(x)\mathbf{F}(\mathbf{u})(x,t)dxdt \\ &- \left(\int_{t^n}^{t^{n+1}} \theta_k(x_{i+1})\tilde{\mathbf{F}}(\tilde{\mathbf{u}})(x_{i+1},t)dt - \int_{t^n}^{t^{n+1}} \theta_k(x_i)\tilde{\mathbf{F}}(\tilde{\mathbf{u}})(x_i,t)dt\right) \\ &+\varepsilon\left(\int_{t^n}^{t^{n+1}} F_d(\theta_k)(x_{i+1})\tilde{\mathbf{u}}(x_{i+1},t)dt - \int_{t^n}^{t^{n+1}} F_d(\theta_k)(x_i)\tilde{\mathbf{u}}(x_i,t)dt\right) \\ &+ \int_{\omega_i}\int_{t^n}^{t^{n+1}} \theta_k(x)\mathbf{S}(x,t)dxdt. \end{aligned} \tag{6}$$

Numerical fluxes

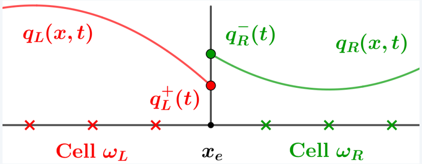
Choice of classical numerical fluxes on edge $x_e = \omega_L \cap \omega_R$

- **Rusanov** (or LLF) for $\tilde{\mathbf{F}}_e(\tilde{\mathbf{u}}_e)$ in the original terms

$$\tilde{\mathbf{F}}_{L/R} = \frac{1}{2}\left[\mathbf{F}(\mathbf{u}_L)^+ + \mathbf{F}(\mathbf{u}_R)^-\right] + \frac{\sigma}{2}\left[\mathbf{u}_L^+ - \mathbf{u}_R^-\right], \tag{7}$$

- **Jump** for $\tilde{\mathbf{u}}_e$ in the extra terms

$$\tilde{\mathbf{u}}_{L/R} = \frac{1}{2}\left[\mathbf{u}_L^+ - \mathbf{u}_R^-\right]\vec{n}_e. \tag{8}$$



ADER framework

ADER methods (**A**rbitrary high order **DER**ivatives) are of Finite Elements type (ADER-FV, ADER-DG) $\Rightarrow$ high order in time = high order in space

- **Prediction step:** determine a space-time Galerkin predictor $\mathbf{q}_i^n(x,t)$ cell by cell

$$\mathbf{q}(x,t) = \sum_{\ell=1}^{L^2} \mathbf{q}_\ell \Theta_\ell(x,t), \tag{9}$$

- **Correction step:** recombine information $\mathbf{u}_i^\star(x)$ between cells + impose BC

$$\mathbf{u}^\star(x) = \sum_{\ell=1}^{L} \mathbf{u}_\ell^\star \theta_\ell(x). \tag{10}$$

ADER framework

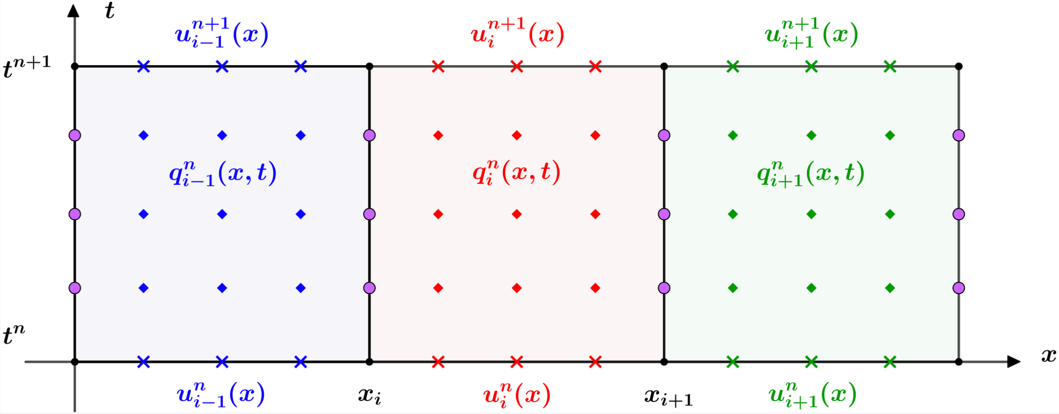


Figure: Sketch of the ADER procedure for a $\mathbb{P}_2$ approximation.

ADER framework

ADER variational formulation for the 1D conservation law $\partial_t \mathbf{q} + \partial_x \mathbf{F}(\mathbf{q}) = \mathbf{S}$

$$\begin{aligned}\int_{\omega_i} \Theta_k(x,t^{n+1})\mathbf{q}(x,t^{n+1})dx - \int_{\omega_i}\int_{t^n}^{t^{n+1}} \partial_t\Theta_k(x,t)\mathbf{q}(x,t)dxdt \\ + \int_{\omega_i}\int_{t^n}^{t^{n+1}} \Theta_k(x,t)\partial_x\mathbf{F}(\mathbf{q})(x,t)dxdt \\ = \int_{\omega_i} \Theta_k(x,t^n)\mathbf{u}^n(x)dx + \int_{\omega_i}\int_{t^n}^{t^{n+1}} \Theta_k(x,t)\mathbf{S}(x,t)dxdt. \end{aligned} \tag{11}$$

with $\mathbb{P}_{L-1}$ expansions $\mathbf{u}(x) = \sum_{\ell=1}^{L} \mathbf{u}_\ell \theta_\ell(x)$ and $\mathbf{w}(x,t) = \sum_{\ell=1}^{L^2} \mathbf{w}_\ell \Theta_\ell(x,t)$.

ADER framework

Summary of the fully explicit, one step, high order **ADER-DG** scheme

1. ⟳ cells: compute the volume source term $\mathbf{S}_C$,
2. ⟳ cells: determine the predictor $\mathbf{q}_C$ s.t.

$$\underline{A}\mathbf{q}_C + \underline{B}\mathbf{F}_C = \underline{C}\mathbf{u}_C^n + \underline{D}\mathbf{S}_C, \tag{12}$$

3. ⟳ edges: compute the wave speeds $\underline{\sigma}_e$ and numerical fluxes $\tilde{\mathbf{u}}_e, \tilde{\mathbf{F}}_e$ (+ BC),
4. ⟳ cells: update the solution $\mathbf{u}_C$... with the DG scheme!

$$\mathbf{u}_C^{n+1} = \mathbf{u}_C^n + \underline{L}\mathbf{F}_C - \left[\underline{M}\tilde{\mathbf{F}}_{C/R} - \underline{N}\tilde{\mathbf{F}}_{L/C}\right] + \varepsilon\left[\underline{P}\tilde{\mathbf{u}}_{C/R} - \underline{Q}\tilde{\mathbf{u}}_{L/C}\right] + \underline{R}\mathbf{S}_C. \tag{13}$$

ADER-DG variants

IPDG methods (**I**nterior **P**enalty **D**iscontinuous **G**alerkin)

$$\int_{\omega_i} \theta_k \mathbf{u}^{n+1} dx = \int_{\omega_i} \theta_k \mathbf{u}^n dx + \int_{t^n}^{t^{n+1}} \left(-\theta_k \tilde{F_d}(\tilde{\mathbf{u}}) + \varepsilon F_d(\theta_k)\tilde{\mathbf{u}}\right)(x_{i+1}, t) dt \pm \cdots \tag{14}$$

Parameter ε	IPDG method	Abbreviation
-1	Symmetric Interior Penalty Galerkin	SIPG
0	Incomplete Interior Penalty Galerkin	IIPG
$+1$	Nonsymmetric Interior Penalty Galerkin	NIPG

Table: Three IPDG methods depending on the value of the parameter ε.

ADER-DG variants

Cattaneo relaxation method based on $\mathbf{w} \to \partial_x \mathbf{u}$ as $\delta = \mathcal{O}(h^L) \to 0$

$$\begin{cases} \partial_t \mathbf{u} + \partial_x (\mathbf{F}_a - \nu \mathbf{w}) = \mathbf{S}, \\ \partial_t \mathbf{w} = \dfrac{1}{\delta} (\partial_x \mathbf{u} - \mathbf{w}). \end{cases} \tag{15}$$

$\Rightarrow$ Hyperbolic system...

$$\partial_t \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} + \partial_x \begin{bmatrix} \partial_{\mathbf{u}} \mathbf{F}_a & -\nu \\ -1/\delta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{S} \\ -\mathbf{w}/\delta \end{bmatrix}, \tag{16}$$

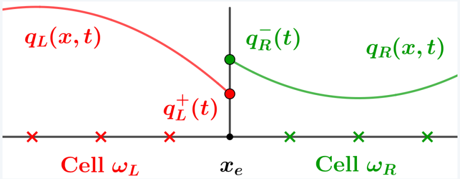
... which is not solved!

ADER-DG variants

Parameters to be chosen to close the method

- Value of $\varepsilon \in \{-1, 0, +1\}$,
- Diffusive wave speed σ_d in the stabilization term $(\sigma = \sigma(\sigma_a, \sigma_d))$

$$\tilde{\mathbf{F}}_{L/R} = \frac{1}{2}\left[\mathbf{F}(\mathbf{q}_L)^+ + \mathbf{F}(\mathbf{q}_R)^-\right] + \frac{\sigma}{2}\left[\mathbf{q}_L^+ - \mathbf{q}_R^-\right]. \tag{17}$$

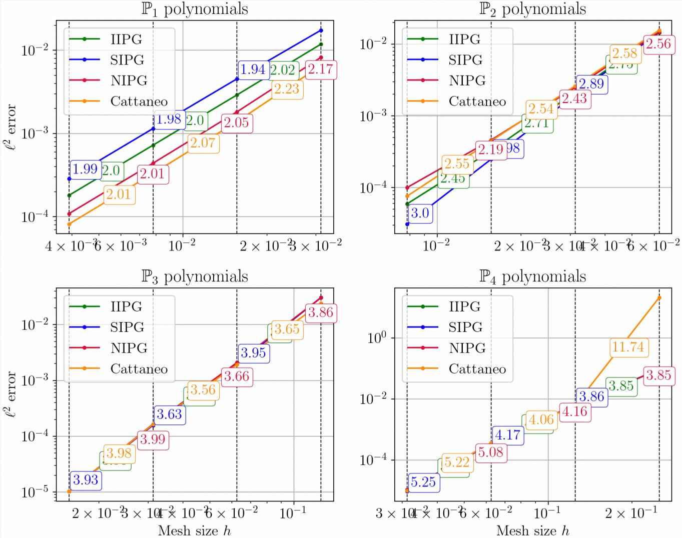


ADER-DG variants

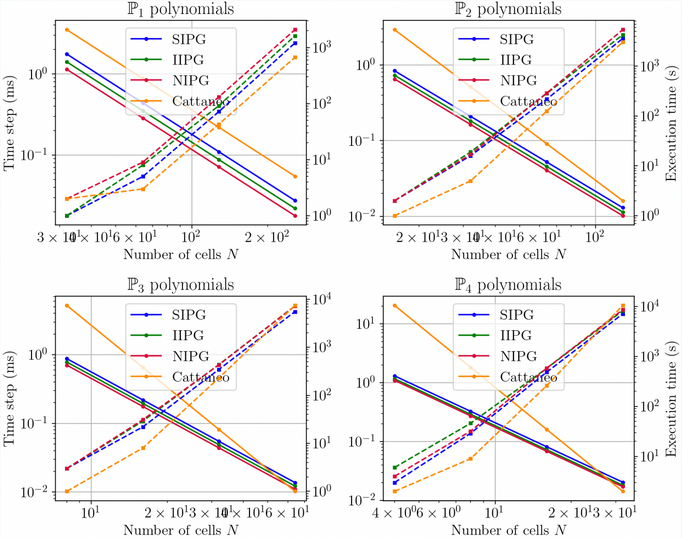
IPDG penalty vs. **Cattaneo** relaxation: an overview

SIPG	IIPG	NIPG	Cattaneo
$\varepsilon = +1$	$\varepsilon = 0$	$\varepsilon = -1$	$\varepsilon = 0$
$\sigma_d = \dfrac{\kappa_{IP}\nu}{h}$			$\sigma_d = \dfrac{\sqrt{\kappa_C \nu}}{h^{L/2}}$
$\kappa_{IP} = \gamma_{IP}(L-1)^2,\ \gamma_{IP} = 10$			$\kappa_C = \gamma_C \left(\dfrac{\sqrt{2}-1}{2^L - \sqrt{2}} \right),\ \gamma_C = 1$
$\sigma = \sigma_a + \sigma_d$			$\sigma = \dfrac{\sigma_a}{2} + \sqrt{\dfrac{\sigma_a^2}{4} + \sigma_d^2}$
Order L	Order $(L-1)$ or L	Order $(L-1)$ or L	Order ?

Numerical results



Numerical results



Conclusion

Two main conclusions

- Among the three IPDG methods: always **S**IPG $\gg$ **I**IPG, **N**IPG

  $\hookrightarrow$ Example: **S** = 1h32, **I** = 1h51, **N** = 2h00 ($N = 64^2, \mathbb{P}_3$).

- **C**attaneo relaxation $\geq$ **S**IPG penalty? Always for $\mathbb{P}_1$ otherwise it depends...

$$\Delta t_{\mathbf{C}}(\nu, L, h) \geq \Delta t_{\mathbf{S}}(\nu, L, h) \quad \Leftrightarrow \quad \frac{\alpha_{\mathbf{C}}(L) \cdot h}{\sigma_{\mathbf{C}}(\nu, L, h)} \geq \frac{\alpha_{\mathbf{S}}(L) \cdot h}{\sigma_{\mathbf{S}}(\nu, L, h)} \quad \Leftrightarrow \quad N \leq N_\star(\nu, L), \qquad (18)$$

$\hookrightarrow N_\star = \infty, 195, 48, 26$ resp. for $\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3, \mathbb{P}_4$ and $\nu = 10^{-2}$,

$\hookrightarrow$ Example: **S** = 8m03 / 1h58, **C** = 4m19 / 2h54 ($N = 16^2/32^2, \mathbb{P}_4$).

$\Rightarrow$ Quick and easy to implement $\oplus$ improved efficiency!

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Thank you for your attention!

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