A moment model for non-equilibrium electrons in a weakly-ionized plasma dominated by electron-neutral collisions

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CANUM,

1. Context

2. Expansion in order of ϵ
   - Expansion of the collision operator
   - Hilbert expansion

3. Moment approximation
   - ϵ^{-1} equation
   - ϵ^{0} equation

Context

Plasma: State of matter with moving electrons, ions, neutrals

Focus on (kinetic) transport of the (light) electrons

Hypothesis:

- Weakly-ionized: $n_n \gg n_e, n_i$
- Parabolic scaling:
  Elastic $n - e$ collisions $\gg$ Transport/Acceleration $\gg$ Other interactions
  $\hookrightarrow$ Isotropizing: $f(v) \to f(|v|)$

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Scaled equation over distribution function $f(t, x, v)$ of electrons:

$$\partial_t f + \frac{v}{\epsilon} \cdot \nabla_x f - \frac{E}{\epsilon} \cdot \nabla_v f = \frac{C_{e,n}(f)}{\epsilon^2} + C(f)$$

with $\epsilon = \sqrt{\frac{m_e}{m_h}} \ll 1$

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Expansion of the collision operator

Expanded elastic e-n collision operator:

$$C_{e,n}(f) = C^0_{e,n}(f) + \epsilon C^1_{e,n}(f) + \epsilon^2 C^2_{e,n}(f) + \mathcal{O}(\epsilon^3)$$

where, writing $v = |v|\omega$,

$$C^0_{e,n}(f) = \int_{\mathcal{S}^2} \left(\tilde{f} - f\right) \nu_{e,n} d\omega, \qquad \tilde{f} = \frac{1}{4\pi} \int_{\mathcal{S}^2} f(|v|\omega) d\omega$$

Property: If $f = \tilde{f}$ is isotropic then

$$C^0_{e,n}(f) = 0.$$

If additionally $f_n = M_n = \frac{n_n}{(2\pi T_n)^{d/2}} \exp(-v_n^2/2T_n)$ then

$$C^1_{e,n}(f) = 0,$$
$$C^2_{e,n}(f) = \frac{1}{v^2} \partial_{|v|} \left[n_n K_{e,n} |v|^3 \left(|v| f + T_n \partial_{|v|} f \right) \right]$$

Hilbert expansion

Distribution function

$$f = f^0 + \epsilon f^1 + \epsilon^2 f^2 + \mathcal{O}(\epsilon^3)$$

Expanded equations:

$$\begin{aligned}
\epsilon^{-2}: \quad & 0 &= C^0_{e,n}(f^0) \\
\epsilon^{-1}: \quad & v \cdot \nabla_x f^0 - E \cdot \nabla_v f^0 &= C^0_{e,n}(f^1) + C^1_{e,n}(f^0) \\
\epsilon^{0}: \quad & \partial_t f^0 + v \cdot \nabla_x f^1 - E \cdot \nabla_v f^1 &= C^0_{e,n}(f^2) + C^1_{e,n}(f^1) + \tilde{C}(f^0) \\
& \tilde{C}(f) &= C^2_{e,n}(f) + C(f)
\end{aligned}$$

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\epsilon^{0}: &\quad & \partial_t f^0 + v\cdot\nabla_x f^1 - E\cdot\nabla_v f^1 &= C^0_{e,n}(f^2) + C^1_{e,n}(f^1) + \tilde{C}(f^0) \\
& & \tilde{C}(f) &= C^2_{e,n}(f) + C(f)
\end{aligned}$$

Implication:

- Order ϵ^{-2} implies $f^0 = \tilde{f}^0$ isotropic (then also $C^1_{e,n}(f^0) = 0$)
- Fredholm alternative: Need

$$\int_{\mathcal{S}^2} C^0_{e,n}(f^1) = 0 = \int_{\mathcal{S}^2} v\cdot\nabla_x f^0 - E\cdot\nabla_v f^0$$

$$\int_{\mathcal{S}^2} C^0_{e,n}(f^2) = 0 = \int_{\mathcal{S}^2} \partial_t f^0 + v\cdot\nabla_x f^1 - E\cdot\nabla_v f^1 - C^1_{e,n}(f^1) - \tilde{C}(f^0)$$

(see also Z. Tazakkati 30/05 12:40 Forban)

ϵ^{-1} equation

Hypothesis: Isotropic $f^0(|v|)$ from the order ϵ^{-2} equation

Weak form of the ϵ^{-1} equation: $\forall \phi \in L^2$

$$\int_{\mathbb{R}^3} \phi \left(v \cdot \nabla_x f^0 - E \cdot \nabla_v f^0 - C^0_{e,n}(f^1) \right) dv \qquad = 0 \qquad (1)$$

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Remark: If $\phi(|v|)$ then (1) holds trivially

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Galerkin approximation: Choose $\phi \in E_N = Span(v, \dots, v|v|^{2N-2})$ then

$$(1) \qquad \Rightarrow \qquad \frac{1}{3} \nabla_x p^{(2l+2)}(f^0) + \frac{2l+3}{3} p^{(2l)}(f^0) E = F^{(2l+1)} \left(C^0_{e,n}(f^1) \right)$$

with the notation

$$p^{(2l)}(f) = \int_v |v|^{2l} f(v) dv, \qquad F^{(2l+1)}(f) = \int_v v|v|^{2l} f(v) dv$$

ϵ^0 equation

Hypothesis: $p^{(2l)}(f) = p^{(2l)}(f^0)$
Weak form of the ϵ^0 equation: $\forall \psi \in L^2$

$$\int_{\mathbb{R}^3} \psi \left(\partial_t f^0 + v \cdot \nabla_x f^1 - E \cdot \nabla_v f^1 - C^1_{e,n}(f^1) - \tilde{C}(f^0) \right) dv \quad = 0 \qquad (2)$$

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Galerkin approximation: Choose $\psi \in E_N = Span(1, \ldots, |v|^{2N})$ then

$$(2) \qquad \Rightarrow \qquad \partial_t p^{(2l)}(f^0) + div_x F^{(2l+1)}(f^1) + 2l F^{(2l-1)}(f^1) \cdot E = p^{(2l)}\left(\tilde{C}(f^0)\right)$$

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Choose $\psi \in E_N = Span(v, \ldots, |v|^{2N-2} v)$ then

$$(2) \qquad \Rightarrow \qquad div_x M^{(2l+2)}(f^1) + 2l M^{(2l)}(f^1) \cdot E = F^{(2l+1)} \left(C^0(f^2) + \tilde{C}(f^0) \right)$$

with the notation

$$M^{(2l)}(f) = \int_v v v^T |v|^{2l-2} f(v) dv$$

Moment system

For all $l < N$:

$$\frac{1}{3}\nabla_x p^{(2l+2)}(f^0) + \frac{2l+3}{3} p^{(2l)}(f^0)E = F^{(2l+1)}\left(C^0_{e,n}(f^1)\right),$$

$$\partial_t p^{(2l)}(f^0) + div_x F^{(2l+1)}(f^1) + 2l F^{(2l-1)}(f^1)\cdot E = p^{(2l)}\left(\tilde{C}(f^0)\right),$$

Remark: Even moments of f^0 and odd moments of f^1

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Remark: Even moments of f^0 and odd moments of f^1

And potentially add

$$div_x M^{(2l+2)}(f^1) + 2l M^{(2l)}(f^1)\cdot E = F^{(2l+1)}\left(C^0(f^2) + \tilde{C}(f^0)\right),$$

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Conclusion

To sum up:

- Hypotheses for the considered physics
  $\hookrightarrow$ Scaled electrons transport equation
  $\hookrightarrow$ Reduced to 1 small parameter ϵ
- Asymptotic expansion with respect to this parameter
  $\hookrightarrow$ Expanded collision operators
  $\hookrightarrow$ Hilbert expansion for the equation
  $\hookrightarrow$ Additional assumptions on f^0 and f^1
- Moment reduction $=$ Galerkin approximation on these equations

Other considerations:

- Moment closure $=$ Solve the Galerkin system
- Entropy considerations through the derivation
- Numerics and physical applications