Estimations d'énergie discrètes pour un modèle hyperbolique d'équations dispersives

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Outline

Framework

Introduction - Context and issues

The LcT model

Annex

Results and outlook

Framework

Collaboration with the SHOM (*Service Hydrographique et Océanographie de la Marine*).

Tolosa : Open source simulation platform for free-surface models, applications in coastal and large-scale oceanography (Shallow Water, multilayer SW and dispersive models).

Contributors :

- ▷ **R. Baraille, M. Ciavaldini, F. Couderc, P. Noble, J.P. Vila** - Toulouse
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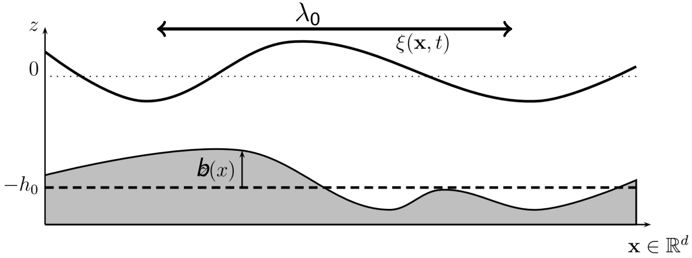
- ▷ **F. Marche** - Montpellier
- ▷ **M. Kazakova, Y. C. Hung** - Chambéry
- ▷ **G.L. Richard** - Grenoble
- ▷ **V. Duchêne** - Rennes

Outline

On dispersive models



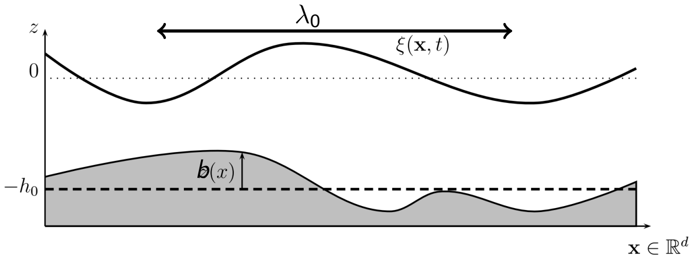
Free surface Euler equations

$$\nabla.\mathbf{v} = 0\,,$$
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\,\mathbf{v} = -\nabla P\,.$$

$\mathbf{v} = \mathbf{v}(\mathbf{x}, z, t) \in \mathbb{R}^{d+1} \times \mathbb{R}^{+}$.

$\Omega_t = \{(\mathbf{x}, z) \in \mathbb{R}^{d+1}\,,\, -h_0 + b(\mathbf{x}) < z < \xi(\mathbf{x}, t)\}$.

On dispersive models



Depth-averaging

Integration along the vertical coordinate + BC :

$$\mathbf{u}(\mathbf{x},t) = \frac{1}{h}\int_{z(\mathbf{x})}^{\xi(\mathbf{x},t)} \mathbf{v}_h(\mathbf{x},z,t)dz\,.$$

- $\mathcal{O}(1)$: Shallow Water
- $\mathcal{O}(\mu)$: Boussinesq, Serre-Green-Naghdi,...

$$\mu = \left(\frac{h_0}{\lambda_0}\right)^2 \ll 1$$

▷ **D. Lannes**, *The Water Waves problem : mathematical analysis and asymptotics*, 2013.

Resulting equations

Shallow Water and SGN equations

$$\begin{cases} \partial_t h + \partial_x(hu) = 0\,, \\ \partial_t hu + \partial_x\Big(hu^2 + \frac{1}{2}gh^2 + \underbrace{\frac{1}{3}h^2\ddot{h} + \Pi}_{\mathcal{O}(\mu)}\Big) = -gh\partial_x b - \underbrace{f}_{\mathcal{O}(\mu)}\,. \end{cases}$$

- Notations

$$\dot{h} = \frac{Dh}{Dt} = \partial_t h + u\partial_x h \quad , \quad \ddot{h} = \frac{D\dot{h}}{Dt}\,.$$

$$\Pi = \frac{h^2}{2}\frac{D[u\partial_x b]}{Dt} \quad , \quad f = h\partial_x b\left(\frac{\ddot{h}}{2} + \frac{D[u\partial_x b]}{Dt}\right)\,.$$

Reformulation of the SGN equations (1)

Elliptic operator

$$\begin{cases} \partial_t h + \partial_x(hu) = 0\,, \\ (I + \mathcal{T}[h,b])\left(\partial_t hu + \partial_x(hu^2)\right) + gh\partial_x\xi + h\mathcal{Q}[h,u,b] = 0\,. \end{cases}$$

$$\mathcal{T}[h,b]w = -\frac{h^2}{3}\partial_x^2 w - h\partial_x h\partial_x w + f(b)w\,.$$

▷ **P. Bonneton *et al.***, *A splitting approach for the fully nonlinear and weakly dispersive Green-Naghdi model*, 2011.

▷ **D. Lannes, F. Marche**, *A new class of fully nonlinear and weakly dispersive Green–Naghdi models for efficient 2D simulations*, 2015.

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$$\mathcal{T}[h,b]w = -\frac{h^2}{3}\partial_x^2 w - h\partial_x h\partial_x w + f(b)w\,.$$

- We set :

$$\mathcal{D} = gh\partial_x\xi - (I + \mathcal{T}[h,b])^{-1}\left(gh\partial_x\xi + h\mathcal{Q}\right)\,.$$

- Shallow Water with source term :

$$\begin{cases} \partial_t h + \partial(hu) = 0\,, \\ \partial_t hu + \partial_x(hu^2) + gh\partial_x\xi = \mathcal{D}\,. \end{cases}$$

▷ **P. Bonneton *et al.***, *A splitting approach for the fully nonlinear and weakly dispersive Green-Naghdi model*, 2011.

▷ **D. Lannes, F. Marche**, *A new class of fully nonlinear and weakly dispersive Green–Naghdi models for efficient 2D simulations*, 2015.

Reformulation of the SGN equations (2)

Hyperbolic problem with constraint

$$\begin{cases} \partial_t h + \partial_x (hu) = 0\,, \\ \partial_t hu + \partial_x \left(hu^2 + \frac{1}{2} g h^2 + \frac{1}{3} h^2 \ddot{h} \right) = 0\,. \end{cases}$$

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- We set $p = h\ddot{h}$.

Reformulation of the SGN equations (2)

Hyperbolic problem with constraint

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- We set $p = h\ddot{h}$.

$$\partial_t hw + \partial_x(huw) = p\,,$$

where $w = -h\partial_x u \quad \rightsquigarrow \quad$ **constraint**.

▷ **E.D. Fernandez-Nieto, M. Parisot, Y. Penel, J. Sainte-Marie**, *A hierarchy of dispersive layer-averaged approximations of Euler equations for free surface flows*, 2018.

▷ **C. Escalante, T. Morales de Luna, M.J. Castro**, *Non-hydrostatic pressure shallow flows : GPU implementation using finite volume and finite difference scheme*, 2018.

▷ **S. Noelle, M. Parisot, T. Tscherpel**, *A class of boundary conditions for time-discrete Green–Naghdi equations with bathymetry*, 2022.

Reformulation of the SGN equations (2)

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- We set $p = h\ddot{h}$.

$$\partial_t hw + \partial_x(huw) = p\,,$$

where $w = -h\partial_x u \quad \rightsquigarrow \quad$ **constraint**.

- The system is rewritten under the form :

$$\partial_t V + \partial_x A(V) = \Psi(p)\,,$$

with $V = (h, u, w)^T$ and $V \in \mathbb{A}_h := \left\{ {}^t(h, u, w) \in L^2(\Omega)\,,\ w = -h\partial_x u \right\}$

▷ **E.D. Fernandez-Nieto, M. Parisot, Y. Penel, J. Sainte-Marie**, *A hierarchy of dispersive layer-averaged approximations of Euler equations for free surface flows*, 2018.

▷ **C. Escalante, T. Morales de Luna, M.J. Castro**, *Non-hydrostatic pressure shallow flows : GPU implementation using finite volume and finite difference scheme*, 2018.

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Reformulation of the SGN equations (3)

Hyperbolic problem with constraint

$$\partial_t h + \partial_x hu = 0\,,$$

$$\partial_t hu + \partial_x \left(hu^2 + \frac{1}{2} g h^2 + \frac{1}{3} h p \right) = 0\,,$$

$$\partial_t hw + \partial_x (huw) = p\,,$$

$$w = -h \partial_x u \qquad \rightsquigarrow \text{constraint}$$

Reformulation of the SGN equations (3)

Hyperbolic system with relaxation term

$$\partial_t h + \partial_x hu = 0\,,$$

$$\partial_t hu + \partial_x \left(hu^2 + \frac{1}{2} g h^2 + \frac{1}{3} h p \right) = 0\,,$$

$$\partial_t hw + \partial_x (huw) = p\,,$$

$$\partial_t hp + \partial_x (hup) = -\lambda \left(w + h \partial_x u \right)\,, \qquad \lambda \gg 1\,.$$

▷ **N. Favrie, S. Gavrilyuk**, *A rapid numerical method for solving Serre–Green–Naghdi equations describing long free surface gravity waves*, 2017.

▷ **C. Escalante *et al.***, *On high order ADER Discontinuous Galerkin schemes for first order hyperbolic reformulations of nonlinear dispersive systems*, 2019.

▷ **G. Richard**, *An extension of the Boussinesq-type models to weakly compressible flows*, 2021.

▷ **J.L. Guermond *et. al.***, *Robust explicit relaxation technique for solving the Green-Naghdi equations*, 2019.

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The *Leucothéa* model (LcT) - G.L. Richard, 2021

1d version, flat bottom

$$
(LcT) \quad \begin{cases}
\partial_t h + \partial_x(hu) = 0\,, \\
\partial_t(hu) + \partial_x(hu^2 + \frac{1}{2}gh^2 + hP) = 0\,, \\
\partial_t(hW) + \partial_x(huW) = \frac{3}{2}P\,, \\
\partial_t(hP) + \partial_x(huP) = -a^2(2W + h\partial_x u)\,.
\end{cases}
$$

a : acoustic speed
W : depth-averaged vertical speed
P : depth-averaged non-hydrostatic pressure

▷ **G. Richard**, *An extension of the Boussinesq-type models to weakly compressible flows*, 2021.

Model features

▶ Hyperbolicity

$$\partial_t V + A(V)\partial_x V = S(V),$$

$$\lambda_{1,2} = u \quad , \quad \lambda_{3,4} = u \pm \sqrt{gh + P + a^2}.$$

Model features

▶ Hyperbolicity

$$\partial_t V + A(V)\partial_x V = S(V)\,,$$

$$\lambda_{1,2} = u \qquad , \qquad \lambda_{3,4} = u \pm \sqrt{gh + P + a^2}\,.$$

▶ Energy

$$\partial_t E + \partial_x \left((E + \frac{1}{2}gh^2 + hP)u \right) = 0\,,$$

$$E = \frac{1}{2}hu^2 + \frac{1}{2}gh^2 + \frac{2}{3}hW^2 + \frac{1}{2a^2}hP^2\,.$$

Model features

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$$\partial_t V + A(V)\partial_x V = S(V)\,,$$

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▶ Energy

$$\partial_t E + \partial_x \left((E + \frac{1}{2}gh^2 + hP)u \right) = 0\,,$$

$$E = \frac{1}{2}hu^2 + \frac{1}{2}gh^2 + \frac{2}{3}hW^2 + \frac{1}{2a^2}hP^2\,.$$

▶ Dispersion relation

$$\frac{h_0^2}{3a^2}\omega^4 - \omega^2 \left(1 + \frac{k^2 h_0^2}{3}\left(1 + \frac{gh_0}{a^2}\right)\right) + k^2 gh_0 = 0\,.$$

▷ $a \to +\infty$: dispersion relation :

$$\omega_{GN}^2(k) = gk^2 h_0 \left(\frac{1}{1 + (kh_0)^2/3} \right)\,.$$

Numerical scheme

Hyperbolic / acoustic splitting

$$\begin{cases} \partial_t h + \partial_x(hu) = 0\,, \\ \partial_t(hu) + \partial_x(hu^2 + \frac{1}{2}gh^2 + hP) = 0\,, \\ \partial_t(hW) + \partial_x(huW) = \frac{3}{2}P\,, \\ \partial_t(hP) + \partial_x(huP) = -a^2(2W + h\partial_x u)\,. \end{cases}$$

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$$\partial_t E + \partial_x \left((E + \frac{1}{2}gh^2)u \right) = 0\,.$$

$$\partial_t E + \partial_x (hPu) = 0\,.$$

Hyperbolic step

Shallow Water with passive transport

$$\begin{cases} \partial_t h + \partial_x(hu) = 0\,, \\ \partial_t(hu) + \partial_x(hu^2 + \frac{1}{2}gh^2) = 0\,, \\ \partial_t(hW) + \partial_x(huW) = 0\,, \\ \partial_t(hP) + \partial_x(huP) = 0\,. \end{cases}$$

$$\partial_t E + \partial_x\left((E + \frac{1}{2}gh^2)u\right) = 0\,.$$

Discrete counterpart of the energy equation :

$$\frac{E_K^{n+1} - E_K^n}{\Delta t} + \frac{\mathcal{G}^{SW}_{K+1/2} - \mathcal{G}^{SW}_{K-1/2}}{\Delta x} \leq 0\,.$$

$$E_K^{n+1} \leq E_K^n - \frac{\Delta t}{\Delta x}\left(\mathcal{G}^{SW}_{K+1/2} - \mathcal{G}^{SW}_{K-1/2}\right)\,.$$

Hyperbolic step

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Discrete counterpart of the energy equation :

$$\frac{E_K^{n+1} - E_K^n}{\Delta t} + \frac{\mathcal{G}_{K+1/2}^{SW} - \mathcal{G}_{K-1/2}^{SW}}{\Delta x} \leq 0\,.$$

$$E_K^{n+1} \leq E_K^n - \frac{\Delta t}{\Delta x}\left(\mathcal{G}_{K+1/2}^{SW} - \mathcal{G}_{K-1/2}^{SW}\right)\,.$$

Requirements :

- **Explicit** methods.
- Inclusion of **topography** terms.
- **Minimize** diffusion.
- Extension in 2d on **unstructured** meshes.

The CPR approach - Continuous frame

Back to the Shallow Water equations

$$\begin{cases} \partial_t h + \partial_x(hu) = 0\,, \\ \partial_t(hu) + \partial_x(hu^2) + h\partial_x\phi = 0\,, \end{cases} \qquad \phi = g(h+b)\,.$$

- Energy equation : $E = \frac{1}{2}hu^2 + \frac{1}{2}gh^2 + gb.$

$$\partial_t E + \partial_x \left(\left(E + \frac{1}{2}gh^2 \right) u \right) = 0\,.$$

▷ **N. Grenier *et al.***, *An accurate low-Mach cheme for a compressible two-fluid model applied to free-surface flows*, 2013.
▷ **M. Parisot, J.P. Vila**, *Centered-potential regularization for the advection upstream splitting method*, 2016.
▷ **F. Couderc *et. al.***, *An explicit asymptotic preserving low Froude scheme for the multilayer shallow water model with density stratification*, 2017.

The CPR approach - Continuous frame

Back to the Shallow Water equations

$$\begin{cases} \partial_t h + \partial_x (h(u - \delta u)) = 0 \,, \\ \partial_t (hu) + \partial_x (hu(u - \delta u)) + h \partial_x \phi = 0 \,, \end{cases} \qquad \phi = g(h + b) \,.$$

- Energy equation : $E = \frac{1}{2} hu^2 + \frac{1}{2} gh^2 + gb.$

$$\partial_t E + \partial_x \left(\left(E + \frac{1}{2} gh^2 \right) (u - \delta u) \right) = - h \partial_x \phi \delta u \,.$$

▷ **N. Grenier *et al.***, *An accurate low-Mach cheme for a compressible two-fluid model applied to free-surface flows*, 2013.
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- Energy equation : $E = \dfrac{1}{2}hu^2 + \dfrac{1}{2}gh^2 + gb.$

$$\partial_t E + \partial_x \left(\left(E + \frac{1}{2}gh^2 \right)(u - \delta u) \right) = \underbrace{-h\partial_x \phi \delta u}_{-\gamma(\partial_x \phi)^2}\,.$$

- Energy dissipation :

$$u^* = u - \delta u \qquad , \qquad \delta u = \gamma \partial_x \phi\,.$$

▷ **N. Grenier *et al.***, *An accurate low-Mach cheme for a compressible two-fluid model applied to free-surface flows*, 2013.
▷ **M. Parisot, J.P. Vila**, *Centered-potential regularization for the advection upstream splitting method*, 2016.
▷ **F. Couderc *et. al.***, *An explicit asymptotic preserving low Froude scheme for the multilayer shallow water model with density stratification*, 2017.

The CPR approach

Continuous level

$$\begin{cases} \partial_t h + \partial_x (h(u - \delta u)) = 0\,, \\ \partial_t (hu) + \partial_x (hu(u - \delta u)) + h\partial_x \phi = 0\,, \end{cases} \qquad \phi = g(h+b)\,.$$

Energy

$$\partial_t E + \partial_x \left(\left(E + \frac{1}{2} g h^2 \right) (u - \delta u) \right) = \underbrace{-h\partial_x \phi \delta u}_{-\gamma(\partial_x \phi)^2}\,.$$

Discrete level $(u^* = \bar{u} - \gamma \delta \phi)$

$$h_K^{n+1} = h_K^n - \Delta t \partial_K (h u^*)$$
$$(hu)_K^{n+1} = (hu)_K^n - \Delta t \partial_K^{up} (u, h u^*) - \Delta t h_K^{n+1} \partial_K^c \phi\,.$$

Energy

$$\frac{E_K^{n+1} - E_K^n}{\Delta t} + \frac{1}{\Delta x} \left(\mathcal{G}_{K+1/2} - \mathcal{G}_{K-1/2} \right) \leq (1-\gamma) \left(\frac{\Delta t}{\Delta x} \right)^2 \bar{h} \left(\delta_K \phi \right)^2$$

Numerical scheme

Hyperbolic / acoustic splitting

$$
\begin{cases}
\partial_t h + \partial_x(hu) = 0\,, \\
\partial_t(hu) + \partial_x(hu^2 + \frac{1}{2}gh^2 + hP) = 0\,, \\
\partial_t(hW) + \partial_x(huW) = \frac{3}{2}P\,, \\
\partial_t(hP) + \partial_x(huP) = -a^2(2W + h\partial_x u)\,.
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\partial_t h = 0\,, \\
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\partial_t(hP) = -a^2(2W + h\partial_x u)\,.
\end{cases}
$$

$$
\partial_t E + \partial_x \left((E + \frac{1}{2}gh^2)u \right) = 0\,. \qquad \partial_t E + \partial_x\,(hPu) = 0\,.
$$

Including dispersion

Acoustic system (flat bottom)

$$\begin{cases} \partial_t h = 0\,, \\ \partial_t (hu) = -\partial_x (hP)\,, \\ \partial_t (hW) = \dfrac{3}{2} P\,, \\ \partial_t (hP) = -a^2 (2W + h\partial_x u)\,. \end{cases}$$

- Energy equation :

$$\partial_t E + \partial_x (hPu) = 0\,.$$

- Discrete counterpart :

$$E_K^{n+1} \leq E_K^n - \frac{\Delta t}{\Delta x} \left(\mathcal{G}^{ac}_{K+1/2} - \mathcal{G}^{ac}_{K-1/2} \right)\,.$$

Acoustic step (1)

Reformulation

$$\begin{cases} \partial_t h = 0\,, \\ \partial_t(hu) = -\partial_x(hP)\,, \\ \partial_t(hW) = \dfrac{3}{2}P\,, \\ \partial_t(hP) = -a^2(2W + h\partial_x u)\,. \end{cases} \Leftrightarrow \begin{cases} \partial_t h = 0\,, \\ \partial_t u = -\dfrac{1}{h}\partial_x(hP)\,, \\ \partial_t W = \dfrac{3}{2}\dfrac{P}{h}\,, \\ \partial_t P = -a^2\left(2\dfrac{W}{h} + \partial_x u\right)\,. \end{cases}$$

Numerical scheme

$$\begin{cases} \dfrac{u_K^{n+1} - u_K^n}{\Delta t} = -\dfrac{1}{h_K}\partial_K^c(hP^{n+1})\,, \\ \dfrac{W_K^{n+1} - W_K^n}{\Delta t} = \dfrac{3}{2}\dfrac{P_K^{n+1}}{h_K}\,, \\ \dfrac{P_K^{n+1} - P_K^n}{\Delta t} = -a^2\left(2\dfrac{W_K^{n+1}}{h_K} + \partial_K^* u\right)\,. \end{cases}$$

Acoustic step (2)

Numerical scheme

$$\begin{cases} \dfrac{u_K^{n+1} - u_K^n}{\Delta t} = -\dfrac{1}{h_K} \partial_K^c (h P^{n+1}), \\ \dfrac{W_K^{n+1} - W_K^n}{\Delta t} = \dfrac{3}{2} \dfrac{P_K^{n+1}}{h_K}, \\ \dfrac{P_K^{n+1} - P_K^n}{\Delta t} = -a^2 \left(2 \dfrac{W_K^{n+1}}{h_K} + \partial_K^* u \right). \end{cases}$$

$$\partial_K^* u = \frac{1}{\Delta x} \left(u_{K+1/2}^* - u_{K-1/2}^* \right) , \quad u_{K+1/2}^* = \bar{u}_{K+1/2} - \beta \frac{\Delta t}{\Delta x} \left[hP \right]_{K+1/2}^n .$$

- Step 1 : **Explicit** resolution of W and P.
- Step 2 : Evolution of u.

Stability under the CFL condition : $\dfrac{\Delta t}{\Delta x} a \leq 1/2$.

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On dispersive properties - Motivation

Equations SGN

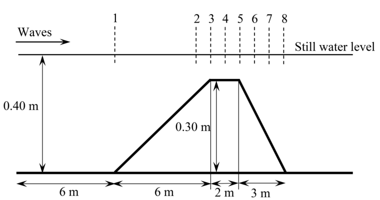
$$\begin{cases} \partial_t h + \partial_x(hu) = 0\,, \\ (I + \mathcal{T}[h,z])\left(\partial_t hu + \partial_x(hu^2)\right) + gh\partial_x \xi + h\mathcal{Q} = 0\,. \end{cases}$$

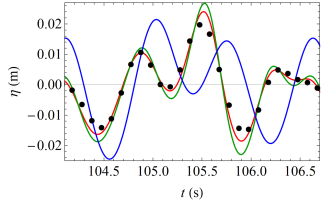
- Dispersion relation :

$$\omega_{GN}^2(k) = gk^2 h_0 \left(\frac{1}{1 + (kh_0)^2/3} \right) .$$

- ω_{GN} *vs.* linear theory (Stokes) : $\omega_S^2(k) = gk\tanh(kh_0)$.

Beji & Battjes test case





▷ **S. Beji, J. Battjes**, *Numerical simulation of nonlinear wave propagation over a bar*, 1994.

Improvement of dispersive properties

Equations SGN

$$\partial_t h + \partial_x hu = 0\,,$$

$$\Big(I + \underbrace{\mathcal{T}[h,z]}_{\mathcal{O}(\mu)}\Big)\left(\partial_t hu + \partial_x(hu^2)\right) + gh\partial_x\xi + \underbrace{h\mathcal{Q}}_{\mathcal{O}(\mu)} = 0\,.$$

▷ **P. Bonneton *et al.***, *A splitting approach for the fully nonlinear and weakly dispersive Green-Naghdi model*, 2011.

Improvement of dispersive properties

Equations SGN

$$\partial_t h + \partial_x hu = 0\,,$$
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- $\partial_t hu = -\partial_x(hu^2) - gh\partial_x\xi + \mathcal{O}(\mu)$.

▷ **P. Bonneton *et al.***, *A splitting approach for the fully nonlinear and weakly dispersive Green-Naghdi model*, 2011.

Improvement of dispersive properties

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- Momentum equation

$$\left(I + \alpha\mathcal{T}[h,z]\right)\left(\partial_t hu + \partial_x(hu^2) + \frac{\alpha-1}{\alpha}gh\partial_x\xi\right) + \frac{1}{\alpha}gh\partial_x\xi + h\mathcal{Q}_1 = 0\,.$$

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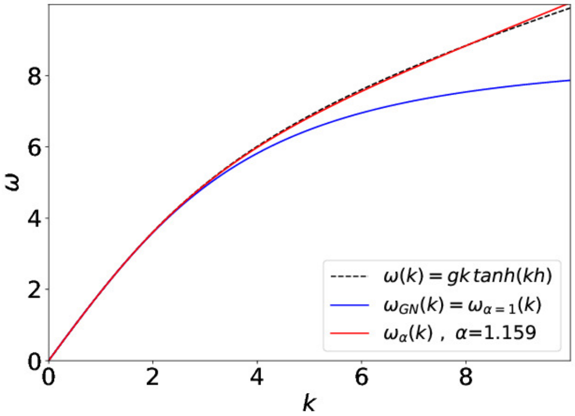
Improvement of dispersive properties

- Dispersion relation :

$$\omega_{\alpha}^2(k) = gk^2 h_0 \left(\frac{1 + (1-\alpha)(kh_0)^2/3}{1 + \alpha(kh_0)^2/3} \right) .$$



▷ **P. Bonneton *et al.***, *A splitting approach for the fully nonlinear and weakly dispersive Green-Naghdi model*, 2011.

The LcT model with improved dispersive properties and exact energy conservation

Final system

$$\begin{cases} \partial_t h + \partial_x (hu) = 0\,, \\ \partial_t (hu) + \partial_x \left(hu^2 + \frac{1}{2} g h^2 + hP\right) = 0\,, \\ \partial_t (hW) + \partial_x (huW) = \frac{3}{2} P + \frac{\alpha - 1}{2\alpha} g h^{3/2} \partial_x B\,, \\ \partial_t (hP) + \partial_x (huP) = -a^2 (2W + \alpha h \partial_x u)\,, \\ \partial_t (hB) + \partial_x (huB) = \partial_x (2 h^{3/2} W)\,. \end{cases}$$

Energy equation

$$\partial_t E + \partial_x \left(\left(E + \frac{1}{2} g h^2 + hP + \Pi_B\right) u \right) = 0\,,$$

$$E = \frac{1}{2} hu^2 + \frac{1}{2} g h^2 + \frac{2}{3\alpha} hW^2 + \frac{1}{2\alpha a^2} hP^2 + \frac{\alpha - 1}{6\alpha^2} g h B^2\,,$$

$$\Pi_B = -\frac{2}{3} \frac{\alpha - 1}{\alpha^2} g h^{3/2} W B\,.$$

Justification of the LcT model

Symmetrizable systems

$$S_0(U)\partial_t U + \sum_{i=1}^{d} S_i(U)\partial_{x_i} U + aL^{\delta}.U = G(U)\,.$$

- Classical frame : control of the solution in H^s.

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- Classical frame : control of the solution in H^s.
- Singular limit problem - three scales : $1, a, \delta$, $(\delta = \sqrt{\mu})$.

Uniform control of :

$$\mathcal{E}_s(U) = \sum_{j=0}^{m} \|\partial_t^j U\|^2_{H^{s-j}} + \sum_{j=m+1}^{s} (a\delta)^{m-j} \|\partial_t^j U\|^2_{H^{s-j}}\,.$$

with respect to $0 < 1/a \leq \delta$.

▷ **V. Duchêne**, *Rigorous justification of the Favrie– Gavrilyuk approximation to the Serre– Green–Naghdi model*, 2019.

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Objectives : (with K. Msheik and V. Duchêne)
- Application to the 2d LcT model with topography.
- Relax conditions on the initial data.

$$\tilde{\mathcal{E}}_s(U) = \sum_{i+j=0}^{s} \alpha_{i,j}^{-2} \|\partial_t^j U\|_{H^i}^2\,.$$

Outline

Perspectives on the Tolosa code

Numerical analysis and schemes

		$\partial_x b = 0$	mild slope	full system
1D	(LcT)	✓	✓	✓
	$(LcT)^{cons}_{\alpha}$	✓	✓	✗
2D NS	(LcT)	✓	(✓)	✗
	$(LcT)^{cons}_{\alpha}$	(✓)	(✓)	✗

Work in progress

- Numerical validations (with F. Couderc).
- Comparaisons SGN vs LcT (with F. Marche).
- Justification of the LcT model (with V. Duchêne, K. Msheik).
- Two-layer extension (with G. Richard, K. Msheik, ...).
- Wave-breaking (PHD of Y. C. Hung, Chambéry).
- High order extension (with L. Emerald, D. Le Roux).
- Multilayer SW (with L. Emerald, P. Noble) .

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