OSWR for the Oseen problem

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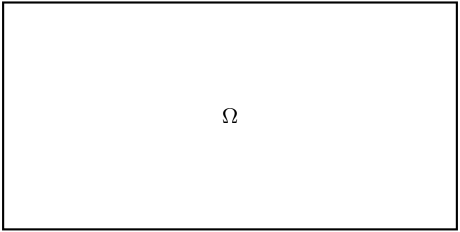
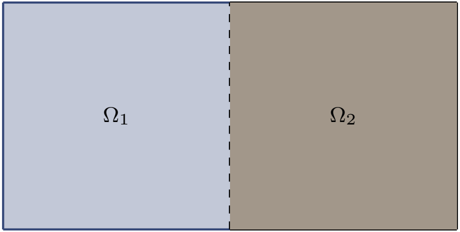


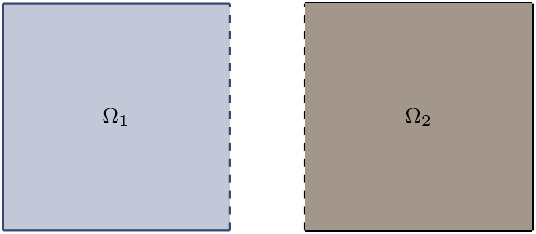
$$\mathcal{L}u = f$$
$$AU = F$$

- With classical methods (e.g. finite volume, finite elements methods) we get a linear system of size $\sim N$.



$$\mathcal{L}u_1 = f \qquad \mathcal{L}u_2 = f$$
$$AU_1 = F \qquad AU_2 = F$$

- With classical methods (e.g. finite volume, finite elements methods) we get two linear systems of size $\sim N/2$.



$$\mathcal{L}u_1 = f \qquad \mathcal{L}u_2 = f$$

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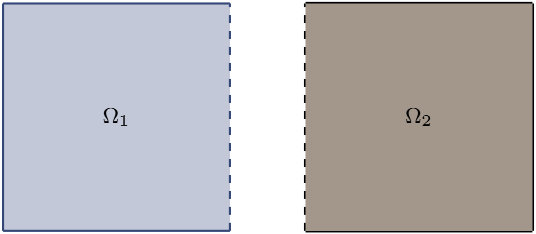
- With classical methods (e.g. finite volume, finite elements methods) we get two linear systems of size $\sim N/2$.
- But unknown boundary condition on the artificial interface



$$\mathcal{L}u_1 = f \qquad \mathcal{L}u_2 = f$$
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- With classical methods (e.g. finite volume, finite elements methods) we get two linear systems of size $\sim N/2$.
- But unknown boundary condition on the artificial interface
- Iterative algorithm

Bibliography

- OSWR Space-time domain decomposition (Gander, Halpern, and Nataf. 2003(OSWR heat 1D), Véronique 2005 (unsteady convection-diffusion))
- Efficient Robin boundary condition with Fourier analysis (Japhet and Nataf 2001, Bennequin et al. 2016 (2D))
- OSWR for the incompressible Stokes (D.-Q. Bui, C. Japhet, and Omnes 2023), OSWR for Oseen, without numerical results (D. Q. Bui 2021)

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Oseen problem

We look for a velocity $\mathbf{u}$ and a pressure p such that

$$\begin{array}{rcll} \mathcal{L}(\mathbf{u},p) := \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + \nabla p & = & \mathbf{f} & \text{in } \Omega\times]0,T[\\ \nabla\cdot\mathbf{u} & = & 0 & \text{in } \Omega\times]0,T[\\ \mathbf{u}(\cdot,0) & = & \mathbf{u}_0 & \text{in } \Omega \\ \mathbf{u} & = & 0 & \text{on } \partial\Omega\times]0,T[\end{array}$$

with given divergence free advection field $\mathbf{b}$, a source term $\mathbf{f}$ and an initial condition $\mathbf{u}_0$. For uniqueness, we state $\int_\Omega p = 0$,

Partition of the domain $\Omega = \Omega_1 \cup \Omega_2$

Equivalent multidomain formulation:

$$\begin{array}{rll} \mathcal{L}(\mathbf{u}_1,p_1) & = \mathbf{f} & \text{in } \Omega_1\times]0,T[\\ \nabla\cdot\mathbf{u}_1 & = 0 & \text{in } \Omega_1\times]0,T[\\ \mathbf{u}_1(\cdot,0) & = \mathbf{u}_0 & \text{in } \Omega_1 \end{array} \qquad \begin{array}{rll} \mathcal{L}(\mathbf{u}_2,p_2) & = \mathbf{f} & \text{in } \Omega_2\times]0,T[\\ \nabla\cdot\mathbf{u}_2 & = 0 & \text{in } \Omega_2\times]0,T[\\ \mathbf{u}_2(\cdot,0) & = \mathbf{u}_0 & \text{in } \Omega_2 \end{array}$$

with physical transmission conditions on $\Gamma\times]0,T[$

$$\mathbf{u}_i = \mathbf{u}_j$$

$$\nu\partial_{\mathbf{n}_{ij}}\mathbf{u}_i\cdot n_{ij} - p_i - \frac{1}{2}(\mathbf{b}_i\cdot\mathbf{n}_{ij})(\mathbf{u}_i\cdot\mathbf{n}_{ij}) = \nu\partial_{\mathbf{n}_{ji}}\mathbf{u}_j\cdot n_{ij} - p_j - \frac{1}{2}(\mathbf{b}_j\cdot\mathbf{n}_{ji})(\mathbf{u}_j\cdot\mathbf{n}_{ji})$$

$$\nu\partial_{\mathbf{n}_{ij}}\mathbf{u}_j\times\mathbf{n}_{ij} - \frac{1}{2}(\mathbf{b}_i\cdot\mathbf{n}_{ij})(\mathbf{u}_j\times\mathbf{n}_{ij}) = \nu\partial_{\mathbf{n}_{ji}}\mathbf{u}_i\times\mathbf{n}_{ji} - \frac{1}{2}(\mathbf{b}_j\cdot\mathbf{n}_{ji})(\mathbf{u}_i\times\mathbf{n}_{ji})$$

Oseen problem

We look for a velocity $\mathbf{u}$ and a pressure p such that

$$\begin{array}{rcll} \mathcal{L}(\mathbf{u},p) \coloneqq \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + \nabla p &=& \mathbf{f} & \text{in } \Omega\times]0,T[\\ \nabla\cdot\mathbf{u} &=& 0 & \text{in } \Omega\times]0,T[\\ \mathbf{u}(\cdot,0) &=& \mathbf{u}_0 & \text{in } \Omega \\ \mathbf{u} &=& 0 & \text{on } \partial\Omega\times]0,T[\end{array}$$

with given divergence free advection field $\mathbf{b}$, a source term $\mathbf{f}$ and an initial condition $\mathbf{u}_0$. For uniqueness, we state $\int_\Omega p = 0$,

Partition of the domain $\Omega = \Omega_1 \cup \Omega_2$

Equivalent multidomain formulation:

$$\begin{array}{rll} \mathcal{L}(\mathbf{u}_1,p_1) &= \mathbf{f} & \text{in } \Omega_1\times]0,T[\\ \nabla\cdot\mathbf{u}_1 &= 0 & \text{in } \Omega_1\times]0,T[\\ \mathbf{u}_1(\cdot,0) &= \mathbf{u}_0 & \text{in } \Omega_1 \end{array} \qquad \begin{array}{rll} \mathcal{L}(\mathbf{u}_2,p_2) &= \mathbf{f} & \text{in } \Omega_2\times]0,T[\\ \nabla\cdot\mathbf{u}_2 &= 0 & \text{in } \Omega_2\times]0,T[\\ \mathbf{u}_2(\cdot,0) &= \mathbf{u}_0 & \text{in } \Omega_2 \end{array}$$

with equivalent Robin transmission condition on boundary $\Gamma\times]0,T[$

$$\begin{array}{rl} \mathcal{B}_1(\mathbf{u}_1,p_1) &= \mathcal{B}_1(\mathbf{u}_2,p_2) \\ \mathcal{C}_1(\mathbf{u}_1) &= \mathcal{C}_1(\mathbf{u}_2) \end{array} \qquad \begin{array}{rl} \mathcal{B}_2(\mathbf{u}_1,p_1) &= \mathcal{B}_2(\mathbf{u}_2,p_2) \\ \mathcal{C}_2(\mathbf{u}_1) &= \mathcal{C}_2(\mathbf{u}_2) \end{array}$$

with

$$\mathcal{B}_i : (\mathbf{u},p) \to \alpha_i\big(\nu(-1)^{i+1}\partial_x\mathbf{u}\cdot\mathbf{n}_i - p - \frac{1}{2}(\mathbf{b}\cdot\mathbf{n}_i)(\mathbf{u}\cdot\mathbf{n}_i)\big) + \mathbf{u}\cdot\mathbf{n}_i$$

$$\mathcal{C}_i : \mathbf{u} \to \beta_i\big(\nu(-1)^{i+1}\partial_x\mathbf{u}\times\mathbf{n}_i - \frac{1}{2}(\mathbf{b}\cdot\mathbf{n}_i)(\mathbf{u}\times\mathbf{n}_i)\big) + \mathbf{u}\times\mathbf{n}_i$$

OSWR algorithm

Given initial Robin data :

for $\ell = 1, 2, \ldots$

$$
\begin{aligned}
\mathcal{L}(\mathbf{u}_1^\ell, p_1^\ell) &= \mathbf{f} && \text{in } \Omega_1 \times]0,T[\\
\nabla \cdot \mathbf{u}_1^\ell &= 0 && \text{in } \Omega_1 \times]0,T[\\
\mathbf{u}_1^\ell(\cdot,0) &= \mathbf{u}_0 && \text{in } \Omega_1 \\
\mathcal{B}_1(\mathbf{u}_1^\ell, p_1^\ell) &= \mathcal{B}_1(\mathbf{u}_2^{\ell-1}, p_2^{\ell-1}) && \text{on } \Gamma \times]0,T[\\
\mathcal{C}_1(\mathbf{u}_1^\ell) &= \mathcal{C}_1(\mathbf{u}_2^{\ell-1}) && \text{on } \Gamma \times]0,T[
\end{aligned}
\qquad
\begin{aligned}
\mathcal{L}(\mathbf{u}_2^\ell, p_2^\ell) &= \mathbf{f} && \text{in } \Omega_2 \times]0,T[\\
\nabla \cdot \mathbf{u}_2^\ell &= 0 && \text{in } \Omega_2 \times]0,T[\\
\mathbf{u}_2^\ell(\cdot,0) &= \mathbf{u}_0 && \text{in } \Omega_2 \\
\mathcal{B}_2(\mathbf{u}_2^\ell, p_2^\ell) &= \mathcal{B}_2(\mathbf{u}_1^{\ell-1}, p_1^{\ell-1}) && \text{on } \Gamma \times]0,T[\\
\mathcal{C}_2(\mathbf{u}_2^\ell) &= \mathcal{C}_2(\mathbf{u}_1^{\ell-1}) && \text{on } \Gamma \times]0,T[
\end{aligned}
$$

OSWR algorithm

Given initial Robin data :

for $\ell = 1, 2, \dots$

$$\begin{array}{rlll rlll}
\mathcal{L}(\mathbf{u}_1^\ell, p_1^\ell) & = \mathbf{f} & \text{in } \Omega_1 \times]0,T[& & \mathcal{L}(\mathbf{u}_2^\ell, p_2^\ell) & = \mathbf{f} & \text{in } \Omega_2 \times]0,T[\\
\nabla \cdot \mathbf{u}_1^\ell & = 0 & \text{in } \Omega_1 \times]0,T[& & \nabla \cdot \mathbf{u}_2^\ell & = 0 & \text{in } \Omega_2 \times]0,T[\\
\mathbf{u}_1^\ell(\cdot,0) & = \mathbf{u}_0 & \text{in } \Omega_1 & & \mathbf{u}_2^\ell(\cdot,0) & = \mathbf{u}_0 & \text{in } \Omega_2 \\
\mathcal{B}_1(\mathbf{u}_1^\ell, p_1^\ell) & = g_{12}^{\ell-1} & \text{on } \Gamma \times]0,T[& & \mathcal{B}_2(\mathbf{u}_2^\ell, p_2^\ell) & = g_{21}^{\ell-1} & \text{on } \Gamma \times]0,T[\\
\mathcal{C}_1(\mathbf{u}_1^\ell) & = \xi_{12}^{\ell-1} & \text{on } \Gamma \times]0,T[& & \mathcal{C}_2(\mathbf{u}_2^\ell) & = \xi_{21}^{\ell-1} & \text{on } \Gamma \times]0,T[
\end{array}$$

Then update the Robin quantities

$$g_{ij}^\ell = \frac{\alpha_{ij}}{\alpha_{ji}} g_{ji}^{\ell-1} - \frac{\alpha_{ij} + \alpha_{ji}}{\alpha_{ji}} \mathbf{u}_j^\ell \cdot \mathbf{n}_{ji}$$

$$\xi_{ij}^\ell = \frac{\beta_{ij}}{\beta_{ji}} \xi_{ji}^{\ell-1} - \frac{\beta_{ij} + \beta_{ji}}{\beta_{ji}} \mathbf{u}_j^\ell \times \mathbf{n}_{ji}$$

We intend to choose the free parameters α_i such as

- the algorithm is well-defined and converges
- the convergence is the fastest possible

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Convergence of the velocity

Convergence

Let $\mathbf{u}_0 \in H^1(\Omega)$, $\mathbf{f} \in L^2(]0,T[,(L^2(\Omega))^2)$. If $\alpha_{ij} = \alpha_{ji}$, $\beta_{ij} = \beta_{ji}$, and if the algorithm is initialized with $g^0_{ij}, \xi^0_{ij} \in L^2(]0,T[,L^2(\Gamma_{ij}))$, then

$$\mathbf{u}_i^\ell \underset{\ell\to\infty}{\longrightarrow} \mathbf{u}|_{\Omega_i}$$

in $\mathcal{C}^0([0,T],(L^2(\Omega_i))^2) \cap L^2(]0,T[,(H^1(\Omega_i))^2)$

Proof is achieved via a weak multidomain formulation and energy estimates.

Non-convergence of the pressure

Relation between Robin quantities:

$$\begin{aligned} g_{12}^{\ell} &= \frac{\alpha_{12}}{\alpha_{21}} g_{21}^{\ell-1} - \frac{\alpha_{12}+\alpha_{21}}{\alpha_{21}} \mathbf{u}_2^{\ell} \cdot \mathbf{n}_2 \\ &= g_{12}^{\ell-2} - \frac{\alpha_{12}+\alpha_{21}}{\alpha_{12}} \mathbf{u}_1^{\ell} \cdot \mathbf{n}_1 - \frac{\alpha_{12}+\alpha_{21}}{\alpha_{21}} \mathbf{u}_2^{\ell} \cdot \mathbf{n}_2 \end{aligned}$$

Divergence free condition:

$$\int_{\partial\Omega_i} \mathbf{u}_i^{\ell} \cdot \mathbf{n}_i = \int_{\Gamma} \mathbf{u}_i^{\ell} \cdot \mathbf{n}_i = 0$$

Finally,

$$\int_{\Gamma} g_{12}^{\ell} = \int_{\Gamma} g_{12}^{\ell-2} = \int_{\Gamma} g_{12}^{0}$$

$\Longrightarrow$ The Robin value on the interface remain constant on odd iterations, and then can not converge to the real Robin value g_{12}

Recovering the pressure

- Based on the technic of D.-Q. Bui, C. Japhet, and Omnes 2023, extended for the Oseen equations, and in two-sided

The error $\mathbf{e}_i^\ell = \mathbf{u}_i^\ell - \mathbf{u}_i$ and $d_i^\ell = p_i^\ell - p_i$

$$\partial_t \mathbf{e}_i^\ell + \Delta \mathbf{e}_i^\ell + (\mathbf{b}\cdot\mathbf{n})(\mathbf{e}_i^\ell \cdot \mathbf{n}) + \nabla d_i^\ell = 0$$

Assume the $\mathbf{e}_i^\ell \to \mathbf{0}$ in $L^\infty(]0,T[,(H^2(\Omega_i))^2)$ and $\partial_t \mathbf{e}_i^\ell \to \mathbf{0}$ in $L^\infty(]0,T[,(L^2(\Omega_i))^2)$, then

$$\nabla d_i^\ell \to 0$$

And with Poincaré–Wirtinger inequality, we get

$$p_i^\ell - (\langle p_i^\ell \rangle_{\Omega_i} - \langle p_i \rangle_{\Omega_i}) \to p_i$$

where $\langle \varphi \rangle_\Omega = \frac{1}{|\Omega|}\int_\Omega \varphi$ is the mean value of φ

$\implies$ we just need to estimate $\langle p_i \rangle_{\Omega_i}$

The error relation for the Robin quantity is

$$\alpha_{ij}\left(\nu\partial_{\mathbf{n}_{ij}}\mathbf{e}_i^\ell\cdot\mathbf{n}_{ij}-\frac{1}{2}(\mathbf{b}\cdot\mathbf{n})(\mathbf{e}_i^\ell\cdot\mathbf{n})-d_i^\ell\right)+\mathbf{e}_i^\ell\cdot\mathbf{n}_{ij}=h_{ij}^{\ell-1}$$

then

$$\|\alpha_{ij}(p_i^\ell-p_i)+g_{ij}^{\ell-1}-g_{ij}\|_{\Gamma_{ij}}\to 0$$

with

- relation $g_{ij}=\frac{\alpha_{ij}}{\alpha_{ji}}g_{ji}-\frac{\alpha_{ij}+\alpha_{ji}}{\alpha_{ij}\alpha_{ji}}\mathbf{u}_j\cdot\mathbf{n}_{ji}$
- divergence free condition
- zero mean condition for pressure $\sum_i p_i=0$
- algebraic calculation

we get a linear system of size $M\times M$ (M being the number of subdomains) such that

$$AY^\ell=B^\ell$$

and finally

$$\tilde{p}_i^\ell:=p_i^\ell-p_i^\ell+Y_i^\ell\xrightarrow[\ell\to\infty]{}p_i$$

At this step, with have defined an algorithm such that

- the velocity converges to the real velocity
- the pressure converges to the real pressure, up to the correction

But for fast convergence, we need to chose efficient Robin parameters α_{ij}, β_{ij}.

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The problem on the error (i.e. with $\mathbf{u}_0 = \mathbf{0}$ and $\mathbf{f} = \mathbf{0}$)

$$\begin{aligned} \partial_t \mathbf{u}_i^\ell - \nu \Delta \mathbf{u}_i^\ell + (\mathbf{b} \cdot \nabla)\mathbf{u}_i^\ell + \nabla p_i^\ell &= 0 \\ \nabla \cdot \mathbf{u}_i^\ell &= 0 \end{aligned}$$

In Fourier decomposition

$$\begin{aligned} \imath\omega u_i^\ell - \nu \partial_{xx} u_i^\ell + \nu k^2 u_i^\ell + b_x \partial_x u_i^\ell + b_y \imath k u_i^\ell + \partial_x p_i^\ell &= 0 \\ \imath\omega v_i^\ell - \nu \partial_{xx} v_i^\ell + \nu k^2 v_i^\ell + b_x \partial_x v_i^\ell + b_y \imath k v_i^\ell + \imath k p_i^\ell &= 0 \\ \partial_x u_i^\ell + \imath k v_i^\ell &= 0 \end{aligned}$$

- We can find explicit solution mode per mode
- With Robin relation on the interface, we get induction relation between solution at the previous step

We get that there exists two matrices $\mathcal{Q}$ and $\mathcal{M}$ (explicitly known) such that

$$\mathbf{u}_{k\omega}^{\ell} = \mathcal{Q}\left(\mathcal{M}(\alpha,\beta,k,\omega)\right)^{\ell}\mathbf{u}_{k\omega}^{0}$$

Then, for minimizing $\mathbf{u}_{k\omega}^{\ell}$, we minimize the eigenvalues of $\mathcal{M}$.

Optimized parameters

$$(\alpha,\beta) := \underset{(\alpha,\beta)\in\mathbb{R}^{+*2}}{\operatorname{argmin}} \max_{(\omega,k)\in[\frac{\pi}{T},\frac{\pi}{\Delta t}]\times[\frac{\pi}{Y},\frac{\pi}{\Delta y}]} \max(|\lambda_1|,|\lambda_2|)$$

where λ_1, λ_2 are the eigenvalues of the 2×2 $\mathcal{M}$ matrix.

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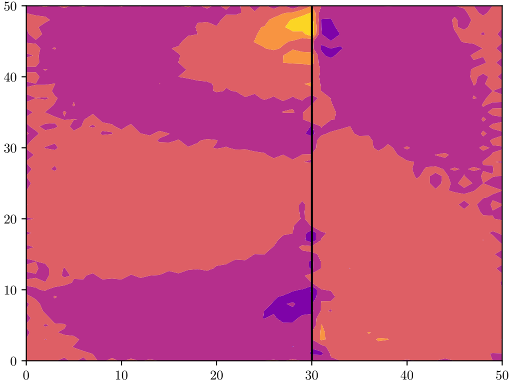
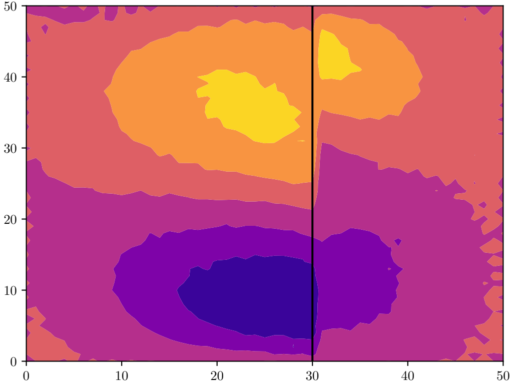
Illustration of the behavior of the convergence of u_x



Iteration 1

$$\mathbf{b} = \begin{pmatrix} \sin(\pi x)\cos(\pi y) \\ -\cos(\pi x)\sin(\pi y) \end{pmatrix}$$

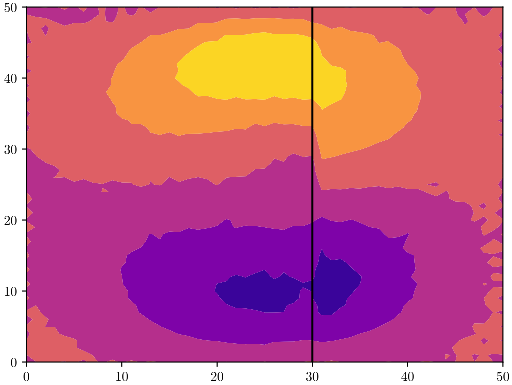
Illustration of the behavior of the convergence of u_x



Iteration 4

$$\mathbf{b} = \begin{pmatrix} \sin(\pi x)\cos(\pi y) \\ -\cos(\pi x)\sin(\pi y) \end{pmatrix}$$

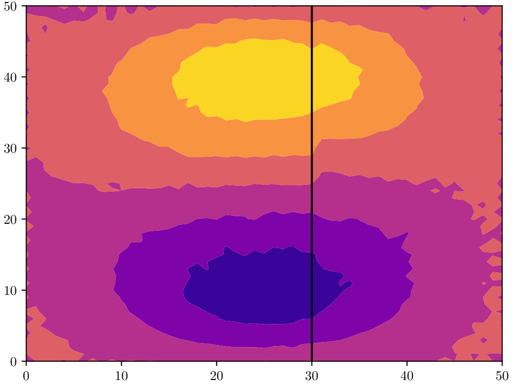
Illustration of the behavior of the convergence of u_x



Iteration 7

$$\mathbf{b} = \begin{pmatrix} \sin(\pi x)\cos(\pi y) \\ -\cos(\pi x)\sin(\pi y) \end{pmatrix}$$

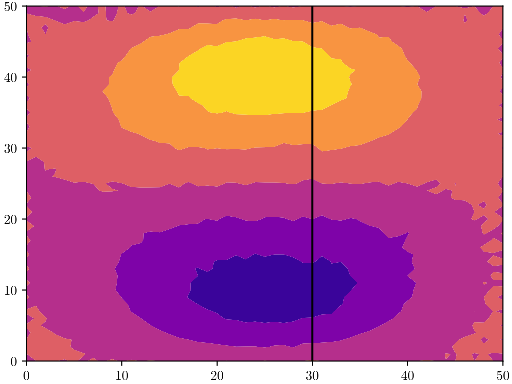
Illustration of the behavior of the convergence of u_x



Iteration 10

$$\mathbf{b} = \begin{pmatrix} \sin(\pi x)\cos(\pi y) \\ -\cos(\pi x)\sin(\pi y) \end{pmatrix}$$

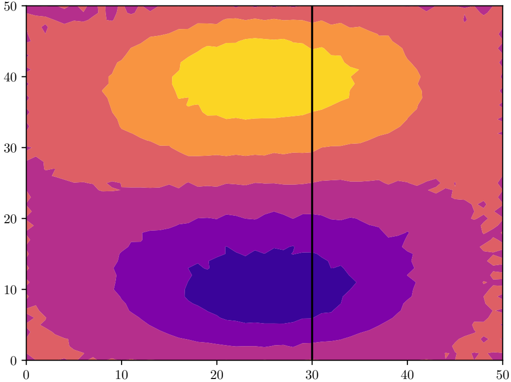
Illustration of the behavior of the convergence of u_x



Iteration 13

$$\mathbf{b} = \begin{pmatrix} \sin(\pi x)\cos(\pi y) \\ -\cos(\pi x)\sin(\pi y) \end{pmatrix}$$

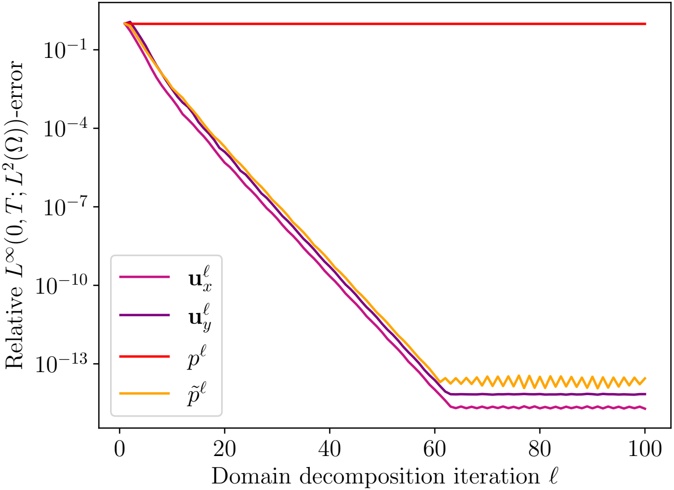
Illustration of the behavior of the convergence of u_x



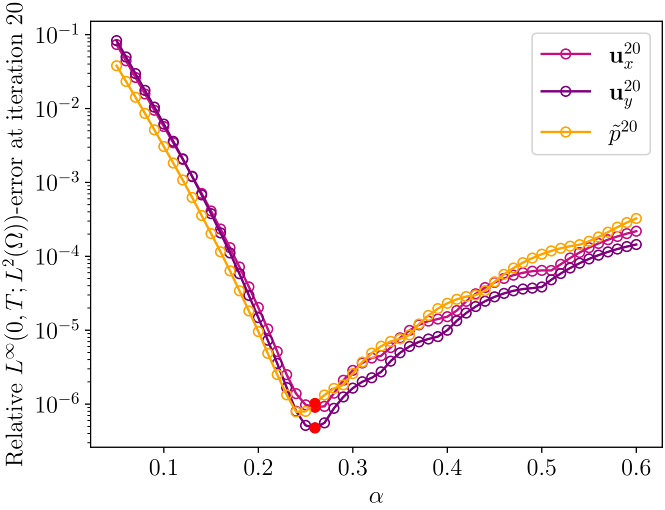
Iteration 16

$$\mathbf{b} = \begin{pmatrix} \sin(\pi x)\cos(\pi y) \\ -\cos(\pi x)\sin(\pi y) \end{pmatrix}$$

Convergence curves



Optimized parameter



Conclusion and future work

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- Convergence of the OSWR algorithm for the velocity
- Convergence with correction for the pressure
- Optimized parameters

Conclusion and future work

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- Convergence of the OSWR algorithm for the velocity
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Current work

- Navier-Stokes
- Coupling time parallel Parareal algorithm

Thank for your attention !

Bibliography

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