A new class of higher-ordered/extended Boussinesq system for efficient numerical simulations by splitting operators

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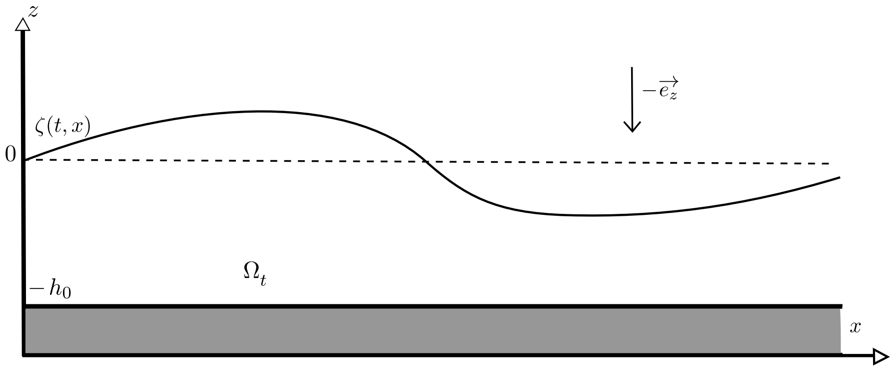
Studying nonlinear, dispersive phenomenon and bathymetric effects in shallow wave dynamics is challenging.

Figure: Tsunami wave train approaching the Japanese coastline after the 2011 Tohoku earthquake (Photo by Douglas Sprott is licensed under CC BY-NC 2.0).

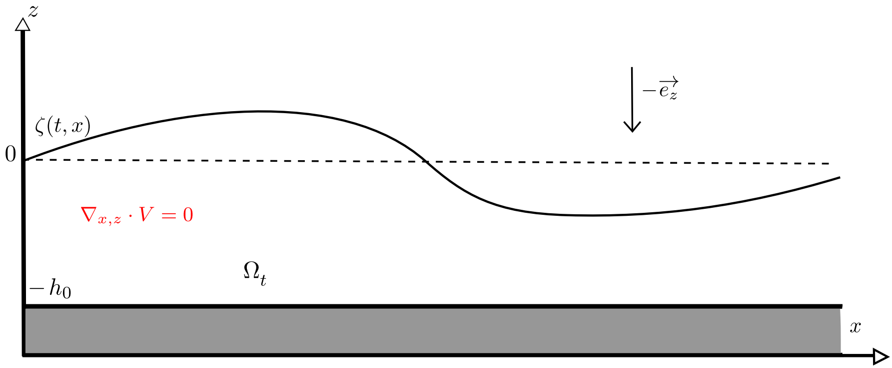
Their understanding is essential to obtain information about extreme wave phenomena such as, storm waves or tsunamis, and what makes them more destructive than other ocean waves.

Governing equations

- **Objective:** Introduce the governing equations of the water-wave problem
- **Means:** Simplifying assumptions on the nature of the fluid.



- Horizontal dimension $d = 1$, flat topography.
- Incompressible, irrotational, inviscid and homogeneous fluid.
- Impenetrable surface and bottom, fluids at rest at infinity.
- There is no surface tension and the external pressure is constant.
- Irrotationality $\Rightarrow V = \nabla_{x,z}\varphi$, (potential) $\Rightarrow \Delta_{x,z}\varphi = 0$ in Ω_t (Laplace equation).



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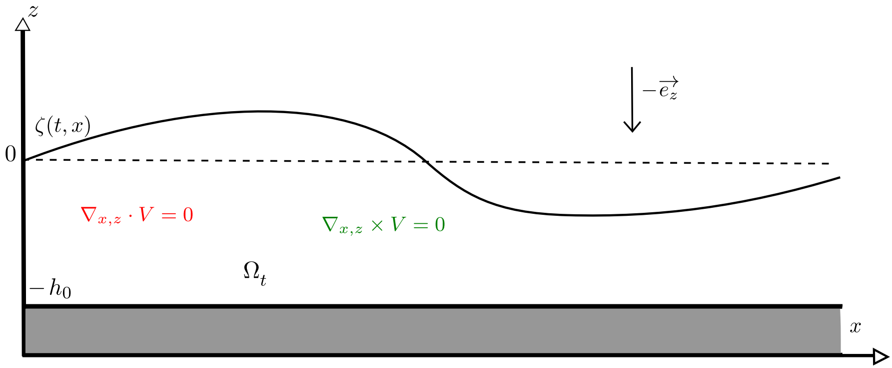

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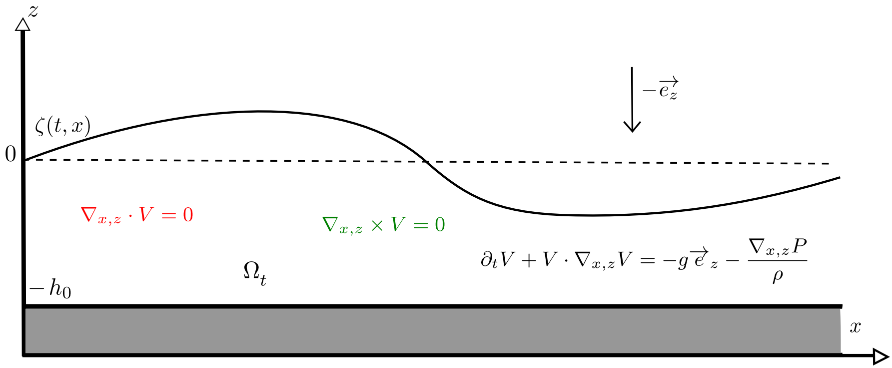

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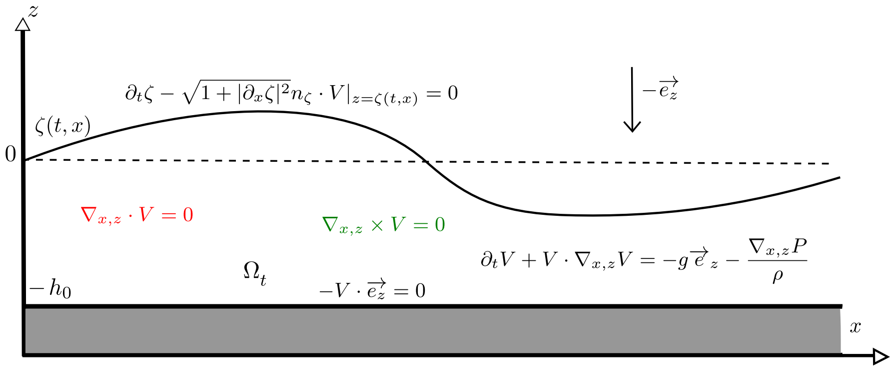

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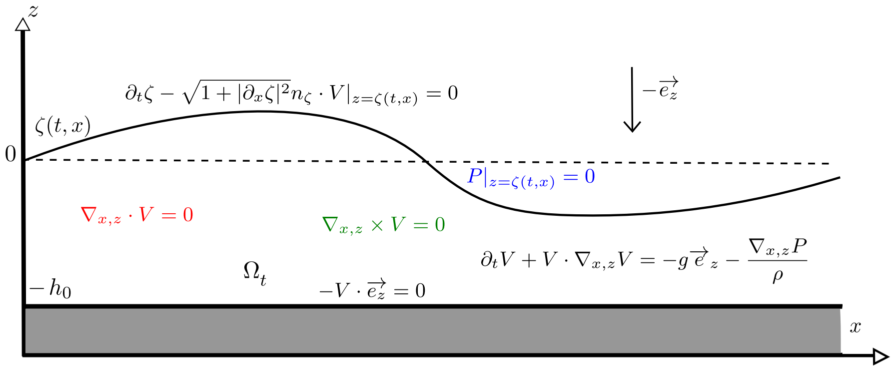

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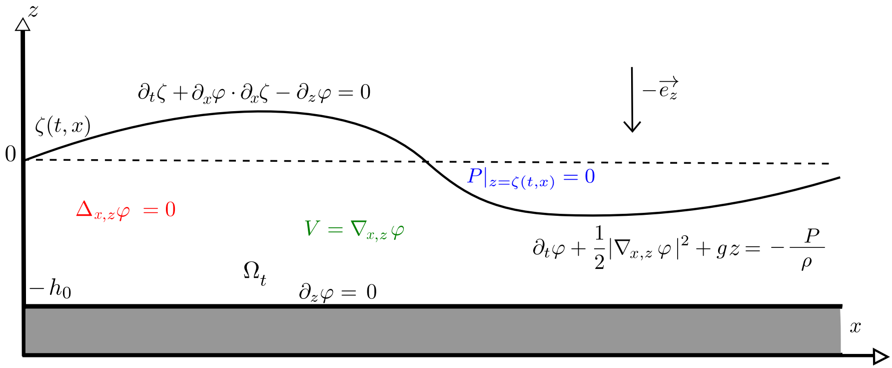

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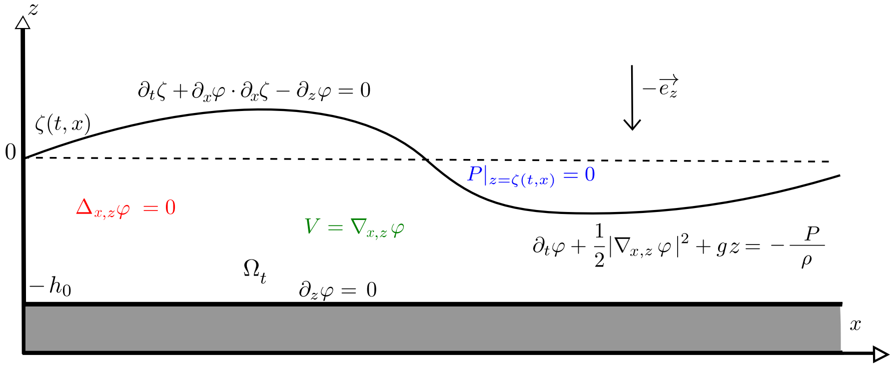
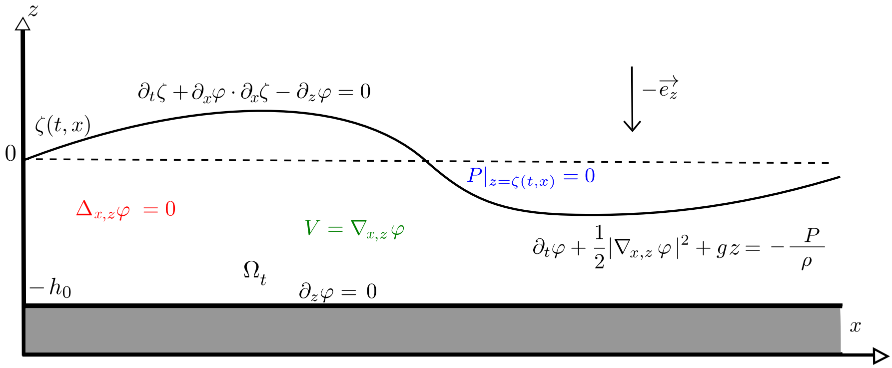


- Large number of equations and unknowns.
- Non linear system of equations.
- Variable domain.



The system can be rewritten as two coupled evolution equations in

$$\zeta \quad \text{and} \quad \psi \equiv \varphi_{|z=\zeta}$$

using Dirichlet-Neumann operators. **[Zakharov '68; Craig, Sulem '93]**

The Zakharov/Craig-Sulem formulation

$$\begin{cases} \partial_t \zeta \; - \; G[\zeta]\psi \; = \; 0, \\ \\ \partial_t \psi \; + \; g\zeta \; + \; \frac{1}{2}|\partial_x \psi|^2 - \frac{\left(G[\zeta]\psi + \partial_x\zeta\partial_x\psi\right)^2}{2(1+|\partial_x\zeta|^2)} = 0. \end{cases} \tag{FE}$$

where the Dirichlet-Neumann operator $G[\zeta]\psi$ is defined by

$$G[\zeta]\psi \equiv \sqrt{1+|\partial_x\zeta|^2}\left(\partial_n \varphi\right)|_{z=\zeta}$$

with φ solution to:

$$\begin{cases} \Delta_{x,z}\varphi = 0 & \text{in } \; \Omega_t, \\ \partial_z \varphi = 0 & \text{on } \{(x,z) \in \mathbb{R}^2, \; z = -h_0\}, \\ \varphi = \psi & \text{on } \{(x,z) \in \mathbb{R}^2, \; z = \zeta\}. \end{cases}$$

- Reduced system of two scalar evolution equations.
- Large-time existence result for the water-waves equations (FE). **[Alvarez-Samaniego, Lannes '08]**
- The description of the solutions from a qualitative and quantitative point of view remains very complex.

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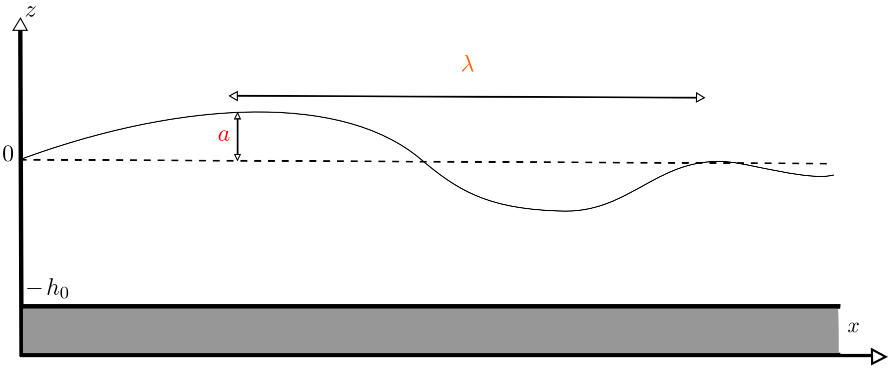
Construction of asymptotic models

- **Objective:** Construct simplified asymptotic models.
- **Means:** Looking for approximation solutions.
- **Difficulties:** Asymptotic expansions of the DN operators.
- **Restrictions:** Dimensionless parameters, specific physical regime.



$$\mu = \frac{{h_0}^2}{\lambda^2}, \quad \varepsilon = \frac{a}{h_0}.$$

Construction of asymptotic models comes from the expansion of the DN operators, w.r.t the shallowness parameter, $\mu \ll 1$ (shallow-water regime).

Dimensionless parameters

A tsunami is an example of wave propagating in a shallow-water regime.
For example, the 2004 Indian ocean tsunami:

Figure: Six weeks after the tsunami hit Banda Aceh on the island of Sumatra, Indonesia (Feb. 12, 2005). `http://www.navy.mil/view_image.asp?id=21836`

$$160 \text{ km} \leq \lambda \leq 240 \text{ km}, \quad 1 \text{ km} \leq h_0 \leq 4 \text{ km} \quad \text{and} \quad a = 60 \text{ cm}$$

The nonlinearity and shallowness parameters are thus estimated by:

$$1.7 \times 10^{-5} \leq \mu \leq 6.2 \times 10^{-4} \quad \text{and} \quad 1.5 \times 10^{-4} \leq \varepsilon \leq 6 \times 10^{-4}$$

- **The Shallow-water regime ($\mu \ll 1$):**
  - **Large amplitude models**: $0 \leq \varepsilon \leq 1$
    - $\longrightarrow$ At first order (w.r.t μ): the *Nonlinear Shallow Water* (or *Saint-Venant*) equations. **[Saint-Venant '1871]**
    - $\longrightarrow$ At second order: the *Green-Naghdi* **[Green, Naghdi '76]**, or *Serre* **[Serre '53]**, or *fully nonlinear Boussinesq*) equations **[Wei *et al.* '95]**.
    - $\longrightarrow$ At third order: the *higher-ordered/extended Green-Naghdi (eGN)*. **[Gobbi '00, Matsuno '15, '16, Khorbatly *et al.* '18]**
  - **Long wave models**: $\varepsilon \sim \mu$
    The *Green-Naghdi* equations can be simplified into the *Boussinesq* equations.
    - $\longrightarrow$ At second order: the *weakly nonlinear Boussinesq* equations. **[Boussinesq '1871, '1872]**
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We are interested in the numerical resolution of the eB equations.

The eB equations

$$\begin{cases} \partial_t\zeta + \partial_x(hv) \;=\; 0, \\ \mathfrak{J}\left(\partial_t v + \varepsilon v\partial_x v\right) + \partial_x\zeta + \dfrac{2}{45}\varepsilon^2\partial_x^5\zeta + +\varepsilon^2\dfrac{2}{3}\partial_x((\partial_x v)^2) = \mathcal{O}(\varepsilon^3)\ , \end{cases}$$

where $h = 1 + \varepsilon\zeta$ is the non-dimensionalised height of the fluid and

$$\mathfrak{J} = 1 + \varepsilon\mathcal{T}[h] - \varepsilon^2\mathfrak{T}\ \text{(coercive)}, \quad \mathcal{T}[h]w = -\frac{1}{3h}\partial_x\left(h^3\partial_x w\right), \quad \mathfrak{T}w = -\frac{1}{45}\partial_x^4 w.$$

- Existence and uniqueness of solution on a relevant time scale of order $1/\sqrt{\varepsilon}$. **[Khorbatly *et al.*'21]**
- This system is more accurate then the 2nd order Boussinesq system containing only weak dispersion.
- Small length comparing to the eGN system.

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Numerical resolution of the eB system

- **Objective:** Numerical resolution, surface waves over flat bottom model.
- **Means:** Suitable reformulation, same order of precision.
- **Novelty:** Improved frequency dispersion, extended range of applicability.
- **Difficulties:** Time dependency, high order derivatives computations.
- **References: [Lannes, Marche '15, Bourdarias, Gerbi, RL '16]** .

Reformulation of the eB model

$$\begin{cases} \partial_t\zeta + \partial_x(hv) = 0, \\ \left(1+\varepsilon\alpha\mathcal{T}[0] - \varepsilon^2\alpha\mathfrak{T}\right)\left(\partial_t v + \varepsilon v\partial_x v + \frac{\alpha-1}{\alpha}\partial_x\zeta\right) + \frac{1}{\alpha}\partial_x\zeta + \frac{7-5\alpha}{45}\varepsilon^2\partial_x^5\zeta \\ +\varepsilon^2\frac{2}{3}\partial_x((\partial_x v)^2) + \varepsilon^2\frac{2}{3}\zeta\partial_x^3\zeta + \varepsilon^2\partial_x\zeta\partial_x^2\zeta = \mathcal{O}(\varepsilon^3). \end{cases} \tag{eB}$$

where $h = 1+\varepsilon\zeta$, $\mathcal{T}[0]w = -\frac{1}{3}\partial_x^2 w$ and $\mathfrak{T}w = -\frac{1}{45}\partial_x^4 w$.

The new reformulation has the same order of precision $\mathcal{O}(\varepsilon^3)$:

- Time-independent invertible operator $\rightarrow$ moderate reduction of computational time.
- A newly added dispersion correction parameter α to improve dispersive properties.

– Does this formulation enjoy improved dispersive properties in large wave numbers regime?

– How to treat high order derivatives that may induce high frequency instability?

Reformulation of the eB model

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$$\begin{cases} \partial_t\zeta + \partial_x(hv) = 0, \\ \left(1 + \varepsilon\alpha\mathcal{T}[0] - \varepsilon^2\alpha\mathfrak{T}\right)\left(\partial_t v + \varepsilon v\partial_x v + \frac{\alpha - 1}{\alpha}\partial_x\zeta\right) + \frac{1}{\alpha}\partial_x\zeta + \frac{7-5\alpha}{45}\varepsilon^2\partial_x^5\zeta \\ +\varepsilon^2\frac{2}{3}\partial_x((\partial_x v)^2) + \varepsilon^2\frac{2}{3}\zeta\partial_x^3\zeta + \varepsilon^2\partial_x\zeta\partial_x^2\zeta = \mathcal{O}(\varepsilon^3). \end{cases} \tag{eB}$$

where $h = 1 + \varepsilon\zeta$, $\mathcal{T}[0]w = -\frac{1}{3}\partial_x^2 w$ and $\mathfrak{T}w = -\frac{1}{45}\partial_x^4 w$.

The new reformulation has the same order of precision $\mathcal{O}(\varepsilon^3)$:

- Time-independent invertible operator $\rightarrow$ moderate reduction of computational time.
- A newly added dispersion correction parameter α to improve dispersive properties.

– Does this formulation enjoy improved dispersive properties in large wave numbers regime?

– How to treat high order derivatives that may induce high frequency instability?

Dispersive properties of the eB model

→ The dispersion relation associated to (eB) is (for plane waves $e^{i(kx-wt)}$):

$$w^2_{\alpha,eB} = \frac{gh_0k^2\left(1+\frac{(\alpha-1)}{3}k^2+\frac{(6-4\alpha)}{45}k^4\right)}{\left(1+\frac{\alpha}{3}k^2+\frac{\alpha}{45}k^4\right)}. \quad (1)$$

→ The exact dispersion relation for the dimenisonalized FE system is:

$$w^2_{F.E} = gh_0|k|\tanh(|k|). \quad (2)$$

→ For $k \ll 1$, the Taylor expansions of (1) and (2) are equivalent provided that $\alpha = 1$.

→ Linear phase and group velocities associated to (1) are defined as:

$$C^p_{eB}(k) = \frac{w_{\alpha,eB}(k)}{|k|} \quad \text{and} \quad C^g_{eB}(k) = \frac{dw_{\alpha,eB}(k)}{dk}.$$

→ To find α_{opt} for $k \in [0,K] \Rightarrow$ Minimize squared relative weighted error:

$$\sqrt{\int_0^K \frac{1}{k}\left(\frac{C^p_{eB}-C^p_S}{C^p_S}+\frac{C^g_{eB}-C^g_S}{C^g_S}\right)^2 dk}. \quad (3)$$

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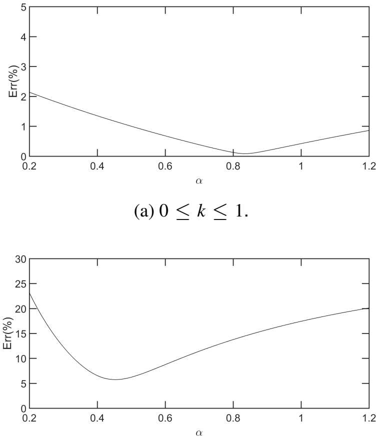

(a) $0 \leq k \leq 1$.

(b) $0 \leq k \leq 2$.

Figure: Phase and group velocities weighted averaged error as a function of α for the (eB) model

- $\rightarrow$ Good dispersion properties in small wave-numbers regime. Error starts to grow rapidly when $k > 1$.
- $\rightarrow$ In large wave-numbers regime (Err=60% for $0 \leq k \leq 10$) with $\alpha = 1$.

Reformulation of the model for numerical resolution

The idea is to factorize the high order derivatives on ζ.
Using the approximation $\partial_x\zeta = (1+\varepsilon\alpha\mathcal{T}[0])^{-1}(\partial_x\zeta) + \mathcal{O}(\varepsilon)$, one gets:

Improved eB formulation with factorized high order derivatives

$$\begin{cases} \partial_t\zeta + \partial_x(hv) = 0, \\ \left(1+\varepsilon\alpha\mathcal{T}[0] - \varepsilon^2\alpha\mathfrak{T}\right)\left(\partial_t v + \varepsilon v\partial_x v + \frac{\alpha-1}{\alpha}\partial_x\zeta\right) + \frac{1}{\alpha}\partial_x\zeta \\ + \frac{7-5\alpha}{45}\varepsilon^2\partial_x^4\left((1+\varepsilon\alpha\mathcal{T}[0])^{-1}(\partial_x\zeta)\right) + \varepsilon^2\frac{2}{3}\partial_x((\partial_x v)^2) \\ +\varepsilon^2\frac{2}{3}\zeta\partial_x^2\left((1+\varepsilon\alpha\mathcal{T}[0])^{-1}(\partial_x\zeta)\right) + \varepsilon^2\partial_x\zeta\partial_x\left((1+\varepsilon\alpha\mathcal{T}[0])^{-1}(\partial_x\zeta)\right) = \mathcal{O}(\varepsilon^3). \end{cases} \tag{eB-f}$$

where $h = 1+\varepsilon\zeta$, $\mathcal{T}[0]w = -\frac{1}{3}\partial_x^2 w$ and $\mathfrak{T}w = -\frac{1}{45}\partial_x^4 w$.

- Avoid the direct calculation of high order derivatives on ζ while keeping the same order of precision $\mathcal{O}(\varepsilon^3)$.
- Extend the range of applicability to high frequency regimes while remaining stable.

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- Avoid the direct calculation of high order derivatives on ζ while keeping the same order of precision $\mathcal{O}(\varepsilon^3)$.
- Extend the range of applicability to high frequency regimes while remaining stable.

The dispersion relation associated to (eB-f) is:

$$\tilde{w}^2_{\alpha,eB-f} = \frac{gh_0k^2\left(1 + \frac{(\alpha-1)}{3}k^2 + \frac{k^4}{45}(\alpha - 1 + \frac{7-5\alpha}{1+\frac{\alpha k^2}{3}})\right)}{\left(1 + \frac{\alpha}{3}k^2 + \frac{\alpha}{45}k^4\right)}. \tag{4}$$

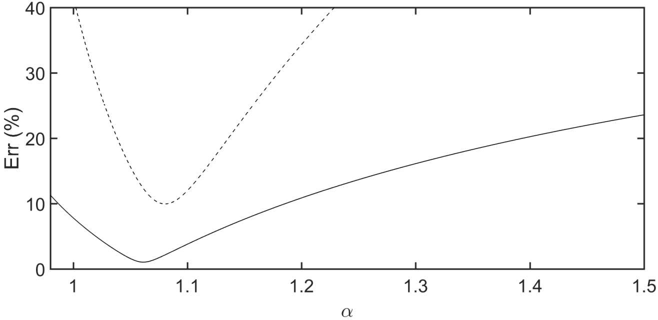


Figure: Phase and group velocities weighted averaged error as a function of α for $0 \leq k \leq 10$. The (eB-f) model is in solid line, the GN-CH model **[Bourdarias, Gerbi, Lteif '16]** is in dots .

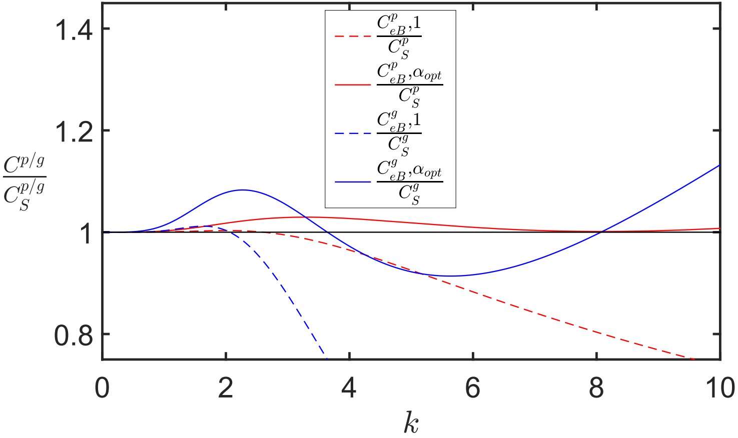
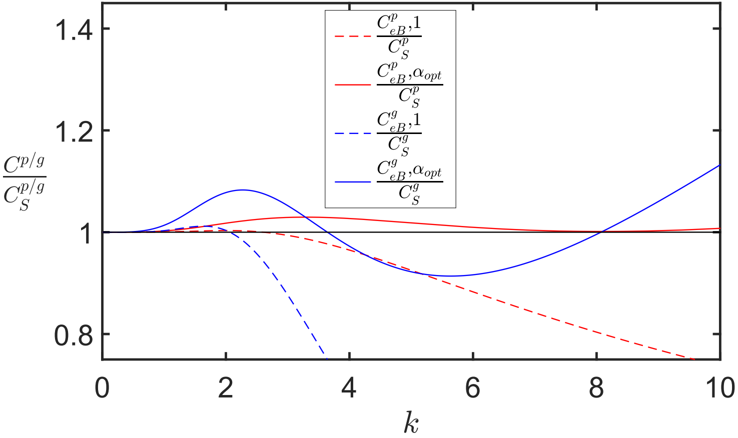


Figure: Errors on linear phase velocity (red) and group velocity (blue). The reference from Stokes theory (black solid line), the (eB-f) model ($\alpha = 1.0610$) in solid lines, the (eB-f) model ($\alpha = 1$) in dashes.

- Significant improvement in the dispersive properties of model (eB-f) with an appropriate choice of α_{opt} in the large frequency regime.



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We investigate the linear behavior of small perturbation to a constant state solution.

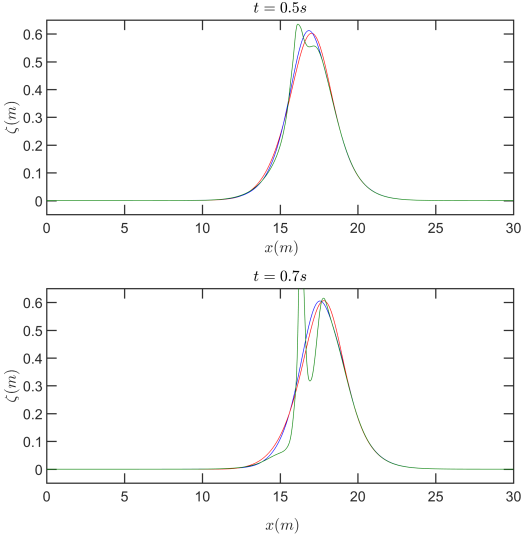

Figure: Comparison at different times between the solutions of the models (eB) (blue line), (eB-f) (red line) and (eB) with just 5th order derivative factorization (green line).

Numerical methods: Splitting scheme

- Strang splitting scheme separating the hyperbolic conservative part and dispersive part of (eB-f). **[Bonneton *et al.* '11, Lannes, Marche '15]**

$$S(\Delta t) = S_1(\Delta t/2)S_2(\Delta t)S_1(\Delta t/2).$$

- $S_1(t)$ is the solution operator related to the hyperbolic NSWE:

$$\begin{cases} \partial_t \zeta + \partial_x(hv) = 0, \\ \partial_t v + \partial_x\left(\frac{\varepsilon}{2}v^2 + \zeta\right) = 0, \end{cases}$$

where $h = 1 + \varepsilon\zeta$.

- $S_2(t)$ is the solution operator related to the dispersive part of the equations.

$$\begin{cases} \partial_t \zeta = 0, \\ \left(1 + \varepsilon\alpha\mathcal{T}[0] - \varepsilon^2\alpha\mathfrak{T}\right)\left(\partial_t v - \frac{1}{\alpha}\partial_x\zeta\right) + \frac{1}{\alpha}\partial_x\zeta \\ + \frac{7-5\alpha}{45}\varepsilon^2\partial_x^4\left((1+\varepsilon\alpha\mathcal{T}[0])^{-1}(\partial_x\zeta)\right) + \varepsilon^2\frac{2}{3}\partial_x((\partial_x v)^2) \\ + \varepsilon^2\frac{2}{3}\zeta\partial_x^2\left((1+\varepsilon\alpha\mathcal{T}[0])^{-1}(\partial_x\zeta)\right) + \varepsilon^2\partial_x\zeta\partial_x\left((1+\varepsilon\alpha\mathcal{T}[0])^{-1}(\partial_x\zeta)\right) = 0, \end{cases}$$

where $\mathcal{T}[0]w = -\frac{1}{3}\partial_x^2 w$ and $\mathfrak{T}w = -\frac{1}{45}\partial_x^4 w$.

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- Spatial discretization of the hyperbolic system using FV method:

  VFRoe method **[Gallouet, Hérard, Seguin '02 '03]** (approximate Godunov scheme). Exact resolution of a linearized Riemann problem.

  - Low dissipation MUSCL reconstruction method is employed **[Camarri *et al.*'04]** (minimize numerical dissipation and dispersion):

$$\mathbf{U}_i^{n,+} = \mathbf{U}_i^n + \frac{1}{2}\delta\mathbf{U}_i^{n,+} \quad \text{and} \quad \mathbf{U}_i^{n,-} = \mathbf{U}_i^n - \frac{1}{2}\delta\mathbf{U}_i^{n,-}.$$

  where:

$$\begin{aligned}\delta\mathbf{U}_i^{n,+} &= \frac{2}{3}(\mathbf{U}_{i+1}^n - \mathbf{U}_i^n) + \frac{1}{3}(\mathbf{U}_i^n - \mathbf{U}_{i-1}^n) - \frac{1}{10}(-\mathbf{U}_{i-1}^n + 3\mathbf{U}_i^n - 3\mathbf{U}_{i+1}^n + \mathbf{U}_{i+2}^n) \\ &\quad - \frac{1}{15}(-\mathbf{U}_{i-2}^n + 3\mathbf{U}_{i-1}^n - 3\mathbf{U}_i^n + \mathbf{U}_{i+1}^n), \\ \delta\mathbf{U}_i^{n,-} &= \frac{2}{3}(\mathbf{U}_i^n - \mathbf{U}_{i-1}^n) + \frac{1}{3}(\mathbf{U}_{i+1}^n - \mathbf{U}_i^n) - \frac{1}{10}(-\mathbf{U}_{i-2}^n + 3\mathbf{U}_{i-1}^n - 3\mathbf{U}_i^n + \mathbf{U}_{i+1}^n) \\ &\quad \frac{1}{15}(-\mathbf{U}_{i-1}^n + 3\mathbf{U}_i^n - 3\mathbf{U}_{i+1}^n + \mathbf{U}_{i+2}^n).\end{aligned}$$

  - To reduce spurious oscillations near discontinuities a limitation procedure is employed.

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  VFRoe method **[Gallouet, Hérard, Seguin '02 '03]** (approximate Godunov scheme). Exact resolution of a linearized Riemann problem.

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$$\mathbf{U}_i^{n,+} = \mathbf{U}_i^n + \frac{1}{2}\delta\mathbf{U}_i^{n,+} \quad \text{and} \quad \mathbf{U}_i^{n,-} = \mathbf{U}_i^n - \frac{1}{2}\delta\mathbf{U}_i^{n,-}.$$

    where:

$$\begin{aligned}\delta\mathbf{U}_i^{n,+} &= \frac{2}{3}(\mathbf{U}_{i+1}^n - \mathbf{U}_i^n) + \frac{1}{3}(\mathbf{U}_i^n - \mathbf{U}_{i-1}^n) - \frac{1}{10}(-\mathbf{U}_{i-1}^n + 3\mathbf{U}_i^n - 3\mathbf{U}_{i+1}^n + \mathbf{U}_{i+2}^n) \\ &\quad - \frac{1}{15}(-\mathbf{U}_{i-2}^n + 3\mathbf{U}_{i-1}^n - 3\mathbf{U}_i^n + \mathbf{U}_{i+1}^n), \\ \delta\mathbf{U}_i^{n,-} &= \frac{2}{3}(\mathbf{U}_i^n - \mathbf{U}_{i-1}^n) + \frac{1}{3}(\mathbf{U}_{i+1}^n - \mathbf{U}_i^n) - \frac{1}{10}(-\mathbf{U}_{i-2}^n + 3\mathbf{U}_{i-1}^n - 3\mathbf{U}_i^n + \mathbf{U}_{i+1}^n) \\ &\quad \frac{1}{15}(-\mathbf{U}_{i-1}^n + 3\mathbf{U}_i^n - 3\mathbf{U}_{i+1}^n + \mathbf{U}_{i+2}^n).\end{aligned}$$

  - To reduce spurious oscillations near discontinuities a limitation procedure is employed.

- Spatial discretization of the dispersive system using FD method:
  - The first, second and fourth order derivatives are discretized using classical fourth order centered formulas.

- Switching values:

FV-FD mix $\Rightarrow$ Classical Taylor expansions of order four are used to switch between the FV unknowns $(\mathbf{U}_i^n)_{i=1,N}$ and the FD unknowns $(\widetilde{\mathbf{U}}_i^n)_{i=1,N+1}$.

- Time discretization:

Classical fourth order Runge-Kutta methods.

- Boundary conditions:

Periodic boundary conditions are treated.

- CFL condition:

CFL-based time step restriction in the first FV sub-step ensures stability for the entire method, overcoming restrictions from high-order derivatives.

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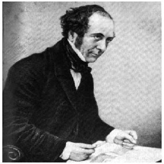

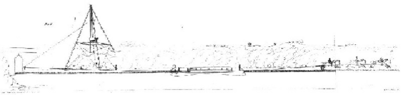

Many pdes have been derived in the literature to model the solitary wave observed by Russell. (scalar: KdV, coupled: Boussinesq and GN). **[Serre '53, Chen '98]**

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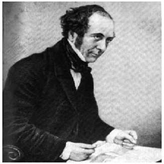

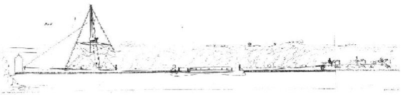

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Numerical validations: Solitary wave solution

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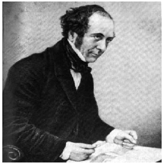

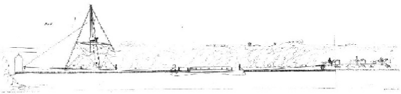

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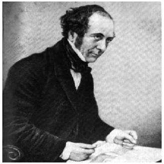

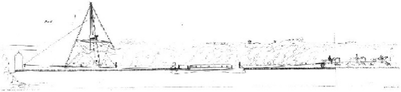

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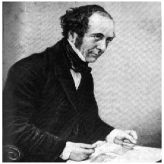

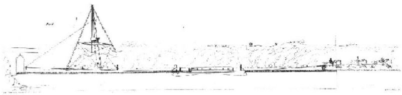

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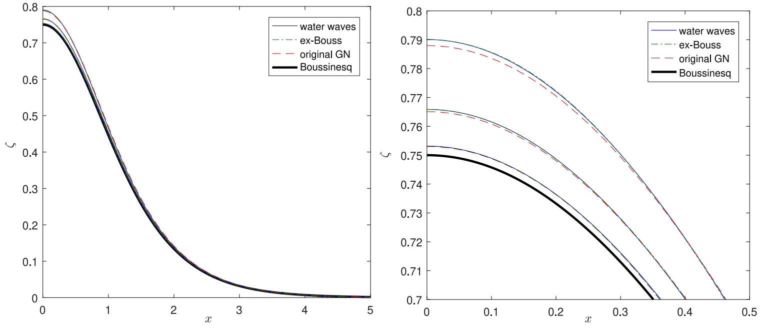
Comparison of the numerical solution of the ODE associated to (eB) **model using the ode45 Matlab Solver with:**
- The water-waves problem (FE) (Matlab script of **[Clamond, Dutykh '13]**).
- The original Green-Naghdi (GN) system, $\zeta_{GN} = (c^2 - 1)\,\text{sech}^2\left(\sqrt{\frac{3(c^2-1)}{4c^2}}\,x\right)$.
- The Boussinesq system, $\zeta_B = \frac{\zeta_{GN}}{c^2}$.



(a) Re-sized waves, $c = 1.025,\ 1.01,\ 1.002$ (b) Zoom in

The (FE) system (water-waves) solution is in better agreement with the solution of the extended Boussinesq model rather than the GN one.

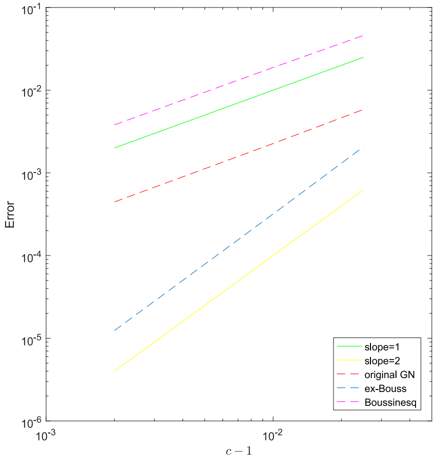


Figure: Normalized l^2-norm of the errors as a function of $c-1$ (log-log plot).

- The (eB) model exhibit a better convergence rate (quadratic) when compared to the original GN model (linear). This highlight the fact that the (eB) model have a better approximate solution.

- Breaking of a regular heap of water with large wave-number:

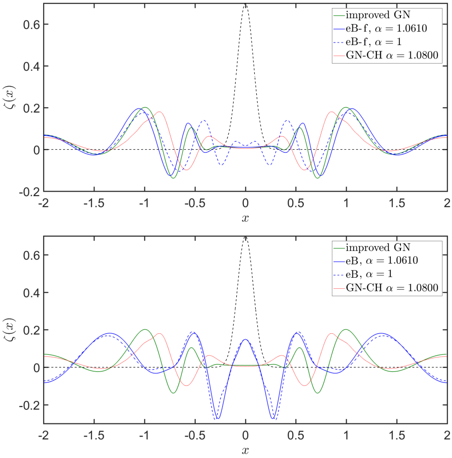


Figure: Comparison of the numerical solutions of (eB-f) model (top) and (eB) model (bottom) with the “improved" GN model and the GN-CH model at $t = 3$.

Factorizing ‘h.o’ derivatives + appropriate $\alpha_{opt} \Rightarrow$ Improved frequency in large wave-numbers regime.

- Dam-break problem:

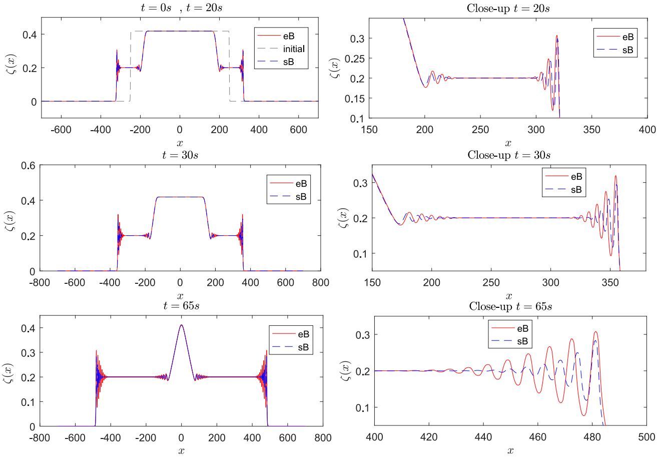


Figure: Dam break: wave shape at different times, comparison between the numerical solution of the (eB) model (solid red line) and sB model (dashed blue lines)

Larger and higher amplitude oscillations of the dispersive tail generated by the eB model due to the 'h.o' nonlinear dispersive terms.

- Favre waves **[Favre '35, Treske '94, Frazao et al. '02]**:

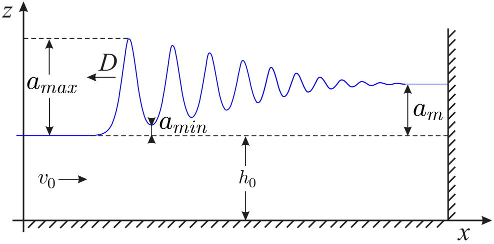


Figure: A sketch of Favre waves.

Due to dispersion, the uniform free surface flow impacting a wall reflects and free surface undulations appear.

A relation between the Froude number and the upstream and downstream water depths can be obtained **[Gavrilyuk '16]**:

$$\frac{h_0 + a_m}{h_0} = \frac{\sqrt{1 + 8F^2} - 1}{2}$$

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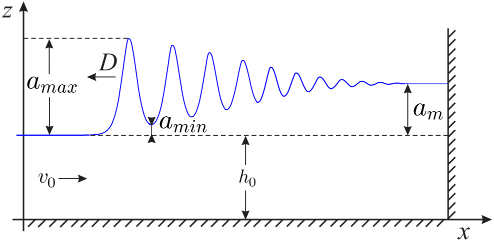

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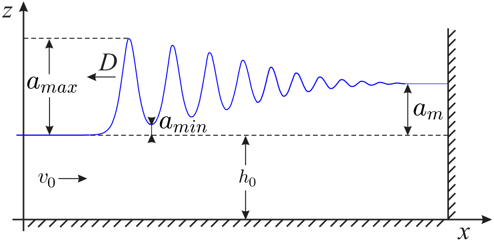

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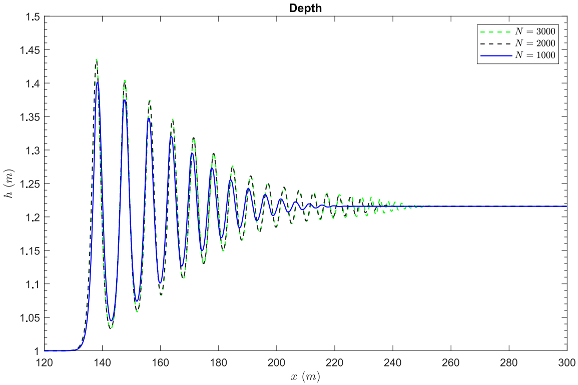
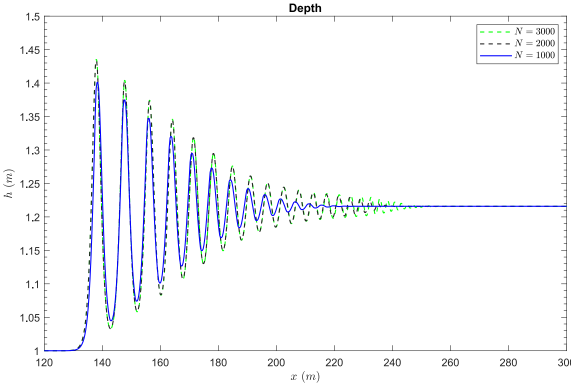


Figure: Comparison of Favre waves obtained by the (eB-f) model at time $t = 54\ s$ for the Froude number $F = 1.16$.

- Convergence is guaranteed.
- The first wave amplitude is well estimated with a finer mesh.
- Accurate prediction of the jump height a_m is provided by the eB model.



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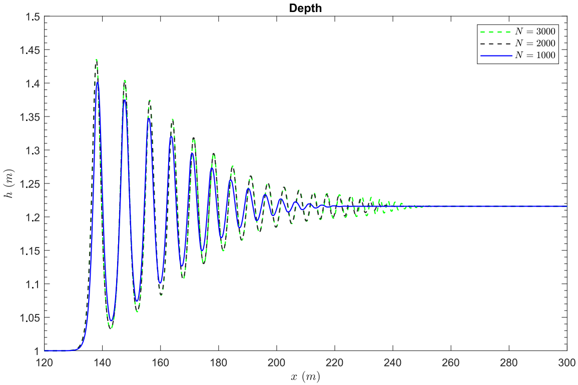

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Comparison of amplitudes of undular bores with experimental data of **[Favre '35]** and **[Trekse '94]**.

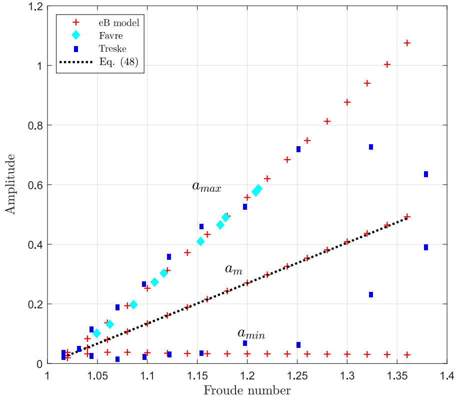

Figure: Amplitude of undular bores for different Froude numbers from the interval $F \in [1.02, 1.36]$

- Good agreement with exp. data until the wave breaking occurs ($F \approx 1.25$). After this critical value our numerical scheme is no more valid since it does not handle wave breaking.

Perspectives:

- The explicit solution of the eB model remains an open problem.
- Numerical resolution of the (eB-f) model in a configuration of variable topography to seriously discriminate high order models.
- Two dimensional extension to study more real-life cases.

Thank you for your attention!