Stabilization of dissipative cat qubits

Rémi Robin,

joint work with Pierre Rouchon and Lev-Arcady Sellem

Quantic team, Mines Paris, Inria, ENS-PSL, CNRS, Université PSL

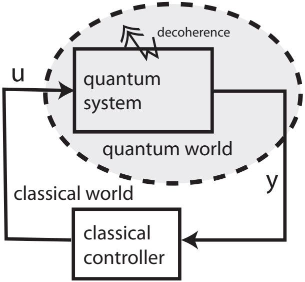
CANUM24, Île de Ré, May 30^{th}, 2024

Toward a quantum computer

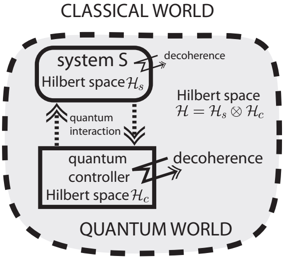
- A qubit is a two-level quantum-mechanical system ($\cong$ unit sphere of $\mathbb{C}^2$)
  - spin of a particle, polarization of a photon . . .
  - a 2 dimensional subset of a higher dimensional system

  $\implies$ We encode a qubit in (a subset of) a physical system.
- Current experiments on qubits : 10^{-3} is the typical error probability during elementary gates (for classical computer $\leq 10^{-18}$). Shor on 2048-bit integer $\sim 10^{12}$ gates.
- Quantum Error Corrections (QEC): Use many physical qubits to encode a single logical qubit. Very costly overhead + threshold.
  $\implies$ Protection against noise is critical and reduction by several orders of magnitude is required.

Two kinds of quantum feedback[1]



Measurement-based feedback: **controller is classical**; measurement back-action on the quantum system of Hilbert space $\mathcal{H}$ is stochastic.



Coherent/autonomous feedback and reservoir/dissipation engineering: the **system of Hilbert space $\mathcal{H}_s$** is coupled to **another quantum system**.

[1] Wiseman/Milburn: Quantum Measurement and Control, 2009, Cambridge University Press.

Time evolution of open quantum systems

Closed system	Open system
$\mathcal{H}$: Hilbert space, $\lvert\psi\rangle \in \mathcal{H}$, $\|\lvert\psi\rangle\| = 1$, $\lvert\psi\rangle \sim e^{i\theta}\lvert\psi\rangle$;	$\rho \in \mathcal{K}^1(\mathcal{H})$, $\rho = \rho^\dagger \geq 0, \mathrm{Tr}(\rho) = 1$
(Schrödinger/Liouville)	(Lindblad)
$\frac{d}{dt}\lvert\psi\rangle = -iH\lvert\psi\rangle \Leftrightarrow \frac{d}{dt}\rho = -i[H,\rho]$	$\frac{d}{dt}\rho = -i[H,\rho] + \sum_\nu D[L_\nu](\rho)$
where $\rho = \lvert\psi\rangle\langle\psi\rvert$ and $[H,\rho] = H\rho - \rho H$	where $D[L](\rho) = L\rho L^\dagger - \frac{1}{2}\left(L^\dagger L\rho + \rho L^\dagger L\right)$

H hermitian operator on $\mathcal{H}$, L_ν: (unbounded) operator on $\mathcal{H}$.

Functional setting

- The set of bounded operators on $\mathcal{H}$ denoted $B(\mathcal{H})$ is a *von Neumann algebra*.
- The predual of $B(\mathcal{H})$ can be identified with the set $\mathcal{K}^1(\mathcal{H})$ of trace-class operators using the trace as duality:

$$\begin{aligned} \mathrm{Tr}: B(\mathcal{H}) \times \mathcal{K}^1(\mathcal{H}) &\to \mathbb{C} \\ (\boldsymbol{X}, \rho) &\mapsto \mathrm{Tr}(\boldsymbol{X}\rho) \end{aligned}$$

- $\mathcal{K}^1(\mathcal{H})$ is a Banach space for the norm

$$\|\rho\|_1 = \mathrm{Tr}\left(|\rho|\right) = \mathrm{Tr}\left(\sqrt{\rho^\dagger \rho}\right)$$

- The dual equation of

$$\frac{d}{dt}\rho = -i[\boldsymbol{H}, \rho] + \sum_j \boldsymbol{L}_j \rho \boldsymbol{L}_j^\dagger - \frac{1}{2}(\boldsymbol{L}_j^\dagger \boldsymbol{L}_j \rho + \rho \boldsymbol{L}_j^\dagger \boldsymbol{L}_j)$$

is

$$\frac{d}{dt}\boldsymbol{X} = i[\boldsymbol{H}, \boldsymbol{X}] + \sum_j \boldsymbol{L}_j^\dagger \boldsymbol{X} \boldsymbol{L}_j - \frac{1}{2}(\boldsymbol{L}_j^\dagger \boldsymbol{L}_j \boldsymbol{X} + \boldsymbol{X} \boldsymbol{L}_j^\dagger \boldsymbol{L}_j).$$

QDS

A quantum dynamical semigroup $(\mathcal{T}_t)_{t\geq 0}$ is a family of operators acting on $B(\mathcal{H})$ which satisfies the following properties:

- $\mathcal{T}_0(\boldsymbol{X}) = \boldsymbol{X}$ for all $\boldsymbol{X} \in B(\mathcal{H})$,
- $\mathcal{T}_{t+s}(\boldsymbol{X}) = \mathcal{T}_t(\mathcal{T}_s(\boldsymbol{X}))$ for all $t, s \geq 0$ and $\boldsymbol{X} \in B(\mathcal{H})$,
- $\mathcal{T}_t(\mathbb{1}) \leq \mathbb{1}$ for all $t \geq 0$,
- $\mathcal{T}_t$ is a completely positive map for all $t \geq 0$. This means that for any finite sequences $(\boldsymbol{X}_j)_{1\leq j\leq n}$ and $(\mathbf{Y}_j)_{1\leq j\leq n}$ of element of $B(\mathcal{H})$, we have

$$\sum_{1\leq j,l\leq n} \mathbf{Y}_l^\dagger \, \mathcal{T}_t(\boldsymbol{X}_l^\dagger \boldsymbol{X}_j) \, \mathbf{Y}_j \geq 0$$

- (normality) for every weakly converging sequence $(\boldsymbol{X}_n)_n \rightharpoonup X$ in $B(\mathcal{H})$, the sequence $(\mathcal{T}_t(\boldsymbol{X}_n))_n$ converges weakly towards $\mathcal{T}_t(\boldsymbol{X})$.
- (ultraweak continuity) for all $\rho \in \mathcal{K}^1$ and $\boldsymbol{X} \in B(\mathcal{H})$, we have

$$\lim_{t\to 0^+} \mathrm{Tr}\left(\rho \mathcal{T}_t(\boldsymbol{X})\right) = \mathrm{Tr}\left(\rho \boldsymbol{X}\right).$$

$\mathcal{L}^*$ is formally the adjoint of $\mathcal{L}$; for $\boldsymbol{X}$ in (a domain in) $B(\mathcal{H})$, it takes the form

$$\mathcal{L}^*(\boldsymbol{X}) = i[\boldsymbol{H}, \boldsymbol{X}] + \sum_j D^*[\boldsymbol{L}_j](\boldsymbol{X}). \tag{1}$$

We introduce $\mathbf{G} = -i\boldsymbol{H} - \frac{1}{2}\sum_j \boldsymbol{L}_j^\dagger \boldsymbol{L}_j$ and assume that $\mathbf{G}$ is the generator of a strongly continuous semigroup of contractions for the Hilbert norm on $\mathcal{H}$.
We say that the quantum dynamical semigroup $(\mathcal{T}_t)_{t\geq 0}$ is solution of Eq. (1) if and only if the following equation is satisfied:

$$\begin{aligned}\langle v|\, \mathcal{T}_t(\boldsymbol{X})\, |u\rangle = \langle v|\, \boldsymbol{X}\, |u\rangle + \int_0^t \Big(\langle v|\, \mathcal{T}_s(\boldsymbol{X})\mathbf{G}\, |u\rangle \\ + \langle v|\, \mathbf{G}^\dagger \mathcal{T}_s(\boldsymbol{X})\, |u\rangle + \sum_j \langle v|\, \boldsymbol{L}_j^\dagger\, \mathcal{T}_s(\boldsymbol{X})\, \boldsymbol{L}_j\, |u\rangle \Big) ds\end{aligned}$$

for all $|u\rangle, |v\rangle \in D(\mathbf{G})$, $\boldsymbol{X} \in B(\mathcal{H})$ and $t \geq 0$.
Under a property known as conservativity of the minimal semigroup, there exists a unique semigroup solution of Eq (1) and we have $\mathcal{T}_t(\mathbb{1}) = \mathbb{1}$ for all $t \geq 0$. In this case, we say that the equation is well-posed and $(\mathcal{T}_t)$ is a **Quantum Markov semigroup**.

Quantum harmonic oscillator

Classical harmonic oscillator

$(x, p) \in \mathbb{R}^2$

$$\frac{d}{dt}x = \omega p = \frac{\partial H}{\partial p}$$
$$\frac{d}{dt}p = -\omega x = -\frac{\partial H}{\partial x}$$

with $H(x, p) = \frac{\omega}{2}(x^2 + p^2)$.

Quantum harmonic oscillator

$\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})$

$$\frac{d}{dt}|\psi\rangle = -iH|\psi\rangle \text{ or } \frac{d}{dt}\rho = -i[H, \rho]$$

with $H = \omega\left(\boldsymbol{X}^2 + \boldsymbol{P}^2\right)$ and
$\boldsymbol{X} = \frac{x}{\sqrt{2}}$, $\boldsymbol{P} = -\frac{i}{\sqrt{2}}\partial_x$.

The hamiltonian H satisfies $\sigma(H) = \omega(\mathbb{N} + \frac{1}{2})$ with a (Fock) basis of eigenstates $(|n\rangle)_{n\in\mathbb{N}}$.
$|0\rangle(x) = \left(\frac{\omega}{\pi}\right)^{1/4} e^{-\frac{\omega}{2}x^2}$

Define $\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} = \frac{x+\partial_x}{\sqrt{2}}$, then $\boldsymbol{a}^\dagger = \boldsymbol{X} - i\boldsymbol{P}$, $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = I$ and
$H = \omega(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{1}{2}I)$.

In the Fock basis : $\boldsymbol{a}|n\rangle = \sqrt{n}|n-1\rangle$, $\boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$,
$\boldsymbol{a}^\dagger\boldsymbol{a}|n\rangle = n|n\rangle$.

Coherent states and cat states

Coherent states

$|\alpha\rangle, \alpha \in \mathbb{C}$

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

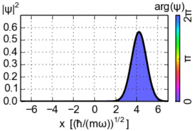
$$e^{-ia^\dagger a\omega t}|\alpha\rangle = \left|e^{i\omega t}\alpha\right\rangle$$

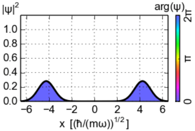
Cat states

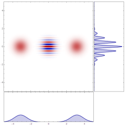
$|+\rangle_\alpha, |-\rangle_\alpha$

$$|+\rangle_\alpha = \frac{|\alpha\rangle + |-\alpha\rangle}{\mathcal{N}_+}$$

$$|-\rangle_\alpha = \frac{|\alpha\rangle - |-\alpha\rangle}{\mathcal{N}_-}$$







Animations from Wikipedia

- Quantum computers are noisy.
- Open quantum systems obey the Lindblad equation.
- The states of a quantum harmonic oscillator : $L^2(\mathbb{R}, \mathbb{C})$ (infinite dimensional).
- Cat states $(\left|+\right\rangle_\alpha, \left|-\right\rangle_\alpha)$.
- Cat qubit: the linear submanifold $\mathrm{Span}(\left|+\right\rangle_\alpha, \left|-\right\rangle_\alpha)$.

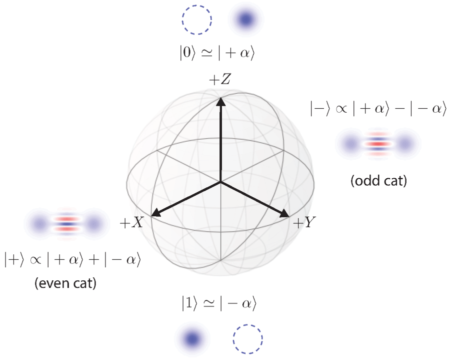


Image from AWS

A first model

Idea : Use dissipation to stabilize the code space $\mathcal{C} = \text{Span}(\left|+\right\rangle_\alpha, \left|-\right\rangle_\alpha) \subset L^2(\mathbb{R}, \mathbb{C})$.

$$\frac{d}{dt}\rho = D[L](\rho) = L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L) \tag{2}$$

with $L = a^2 - \alpha^2 \, \text{Id}$.

Theorem (Azouit, Sarlette, Rouchon, 2016)

For every density operator ρ_0 smooth enough, Equation (2) is well posed and there exists ρ_∞ with support on $\mathcal{C}$ such that $\rho(t) \xrightarrow[t\to\infty]{} \rho_\infty$.

Main idea of the proof

$\text{Tr}(L^\dagger L\rho(t))$ is a strict Lyapunov function.

Formally

$$\begin{aligned}\frac{d}{dt}(L^\dagger L) &= D^*[L](L^\dagger L)\\ &= L^\dagger L^\dagger L L - \frac{1}{2}\left(L^\dagger L L^\dagger L + L^\dagger L L^\dagger L\right)\\ &= L^\dagger [L^\dagger, L] L.\end{aligned}$$

Besides

$$[L^\dagger, L] = [(a^\dagger)^2 - \alpha^2, a^2 - \alpha^2] = [(a^\dagger)^2, a^2] = -2a^\dagger a - 2.$$

Thus formally,

$$\frac{d}{dt}\,\mathrm{Tr}\left(L^\dagger L \rho_t\right) \leq -2\,\mathrm{Tr}\left(L^\dagger L \rho_t\right).$$

Interest

Important features of dissipative cat qubits

- Typical noise in the cavity : $\epsilon_a D[a]$, $\epsilon_{th} D[a^\dagger]$, $\epsilon_d D[a^\dagger a]$...
- Bit-flips ($|0\rangle \approx |\alpha\rangle \to |1\rangle \approx |-\alpha\rangle$) are exponentially suppressed in $|\alpha|^2$
- Nevertheless, phase-flips ($|+\rangle_\alpha \to |-\rangle_\alpha$) increases linearly in $|\alpha|^2$.

$\Longrightarrow$ bias noise.

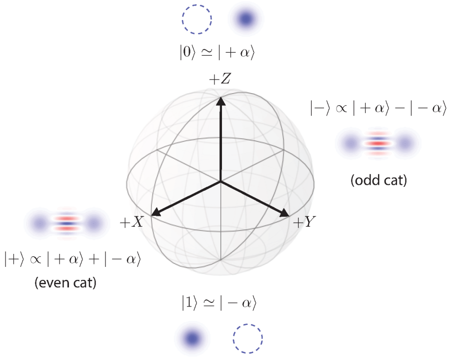


Image from AWS

How can we engineer the dissipator $D[a^2 - \alpha^2]$?

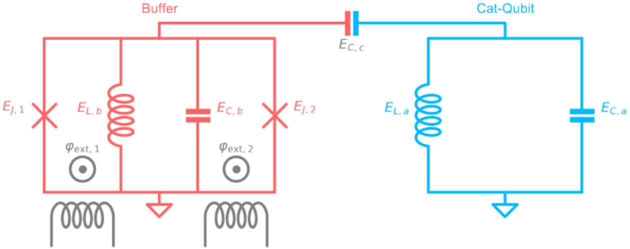
Main tool is reservoir engineering, based on an hamiltonian coupling with a quantum controller system.

- Use a second (lossy) mode denoted by $\mathcal{H}_b \implies \mathcal{H} = L^2(\mathbb{R}, \mathbb{C}) \otimes L^2(\mathbb{R}, \mathbb{C})$.
- Engineer the hamiltonian coupling $L \otimes b^\dagger + L^\dagger \otimes b$.

We obtain the equation

$$\frac{d}{dt}\rho = -ig[Lb^\dagger + L^\dagger b, \rho] + \kappa_b D[b](\rho)$$

Using adiabatic elimination (T fixed, Dim($\mathcal{H}$) $< \infty$, $\kappa_b \to \infty$), we retrieve a reduced system on $\mathcal{H}_a$ with $\frac{g^2}{4\kappa_b} D[L]$.



Exponential suppression of bit-flips in a qubit encoded in an oscillator, R. Lescanne et al, 2020, Nature Physics.

Theorem (R., Rouchon, Sellem, 2023)

For any $g, \kappa_b > 0$, and any density operator ρ_0, the equation

$$\begin{cases} \frac{d}{dt}\rho & = -ig[Lb^\dagger + L^\dagger b, \rho] + \kappa_b D[b](\rho) \\ \rho(t=0) & = \rho_0 \end{cases} \tag{3}$$

with $L = a^2 - \alpha^2 \, \mathrm{Id}$ is well-posed and there exists ρ_∞ with support on $\mathcal{C} \otimes |0\rangle$ such that

$$\rho(t) \xrightarrow[t\to\infty]{} \rho_\infty$$

Tool: A LaSalle's like invariance principle

Assume

- (Tightness) for any density operator ρ_0 and $\epsilon > 0$, there exists a finite dimensional linear manifold E such that the orthogonal projector P_E on E satisfies $\mathrm{Tr}\left(P_E \rho_t\right) > 1 - \epsilon$ for any $t \geq 0$.
- (Density) The span of

$$\left\{ P(G^\dagger, b^\dagger) \left|v\right\rangle \otimes \left|0\right\rangle \mid \text{P non-commutative polynomial, } v \in \mathrm{Ker}\left(L\right) \right\}$$

  is dense ($G = -igH - \frac{\kappa_b}{2} b$).

then

$$\mathrm{Tr}\left(\Pi_L \rho(t)\right) \xrightarrow[t \to \infty]{} 1$$

with $\Pi_L = \Pi_{\mathrm{Ker}(\boldsymbol{L})} \otimes \left|0\right\rangle \left\langle 0\right|$.

Ideas of proof of the new Lasalle principle

Using the density assumption, we can prove that:
For any $t > 0$, there exist a positive self-adjoint operator $\mathbf{S} \geq 0$ such that

$$\begin{cases} \mathcal{T}_t(\Pi_L) \geq \Pi_L + \mathbf{S}, \\ \mathbf{S}|_{\mathcal{H}_L} = 0, \quad \mathbf{S}|_{\mathcal{H}_L^\perp} > 0. \end{cases} \tag{4}$$

($\mathcal{H}_L = \mathrm{Ker}\,(\boldsymbol{L}) \otimes |0\rangle$.)

The main tool is the following integral representation formula: for any $|u\rangle \in D(\mathbf{G}^\infty)$,

$$\langle u|\, \mathcal{T}_t(\Pi_L)\, |u\rangle = \langle u|\, e^{t\mathbf{G}^\dagger}\, \Pi_L\, e^{t\mathbf{G}}\, |u\rangle + \kappa \int_0^t \langle u|\, e^{(t-s)\mathbf{G}^\dagger}\, \boldsymbol{b}^\dagger \mathcal{T}_s(\Pi_L)\, \boldsymbol{b}\, e^{(t-s)\mathbf{G}}\, |u\rangle\, ds$$

Tightness

- Introduce $V = \left(\frac{a^\dagger a}{2} + b^\dagger b\right)^2$

$$\mathcal{L}^*(V) = ig\frac{a^\dagger a}{2}(b^\dagger - b) + ig2b^\dagger b(b^\dagger - b) + \kappa_b b^\dagger b - \kappa_b \frac{a^\dagger a}{2} b^\dagger b - 2\kappa_b (b^\dagger b)^2$$

- Adding μW with $W = \mathcal{L}^*(a^\dagger a) = 2i(a^2 b^\dagger - (a^\dagger)^2 b)$, we can prove that there exist $C_1, C_2 > 0$ such that

$$\mathcal{L}^*(V + \mu W) \leq C_1 - C_2(V + \mu W)$$

- This (with some functional analysis) shows that if $\mathrm{Tr}(V\rho_0) < \infty$, then $\sup_{t\in\mathbb{R}^+} \mathrm{Tr}\left(\rho(t)V\right) < \infty$.

Density

Much harder...
A key element is the fact that

$$\mathrm{Span}\left\{(L^\dagger)^j\,|v\rangle \mid j \in \mathbb{N},\ v \in \mathrm{Ker}\,L\right\} \oplus \mathrm{Span}\left\{(L^\dagger)^j[L, L^\dagger]\,|v\rangle \mid j \in \mathbb{N}\,, v \in \mathrm{Ker}\,L\right\}$$

is dense in $L^2(\mathbb{R}, \mathbb{C})$.
Tools :

- Segal–Bargmann representation (Holomorphic function in $L^2(\mathbb{C}, e^{-|z|^2}dz)$).
- Coherent states are the reproducing kernels of this space.
- Decomposition of the space based on zeros of some holomorphic functions using a theorem of Newman and Shapiro.

Extension and futur works

The proof works for 'multi-legged cat' ($L = a^k - \alpha^k$).

Some open questions

- Extension to several dissipators (multiple ancillae, time dependent non-resonant).
- Exponential stability ? Rigorous perturbation analysis.

A few references

- Rémi Robin, Pierre Rouchon, and Lev-Arcady Sellem. *Convergence of bipartite open quantum systems stabilized by reservoir engineering*. 2023. DOI: 10.48550/arXiv.2311.10037
- Mazyar Mirrahimi et al. "Dynamically protected cat-qubits: a new paradigm for universal quantum computation". In: *New Journal of Physics* 16.4 (2014), p. 045014. DOI: 10.1088/1367-2630/16/4/045014
- Rémi Azouit, Alain Sarlette, and Pierre Rouchon. "Well-posedness and convergence of the Lindblad master equation for a quantum harmonic oscillator with multi-photon drive and damping". In: *ESAIM: Control, Optimisation and Calculus of Variations*. Special Issue in honor of Jean-Michel Coron for his 60th birthday 22.4 (2016), pp. 1353–1369. DOI: 10.1051/cocv/2016050
- Franco Fagnola. "Quantum Markov Semigroups and Quantum Flows". In: *Proyecciones* 3 (1999). DOI: 10.22199/S07160917.1999.0003.00004