A robust preconditioner for saddle-point problems in an industrial context

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Outline

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Introduction

Industrial context

PolyMAC

Numerical Resolution

Saddle point problems

Structure of linear systems that is encountered in many different fields:

$$\begin{pmatrix} A & B^T \\ -B & C \end{pmatrix}$$

with

- $A_S = \frac{1}{2}(A + A^T)$ SPD (hereafter, A SPD)
- B full rank
- C SPsD (hereafter $C = 0$)

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Examples:

- (Navier-)Sokes equations
- Constrained optimisation problems
- linear elasticity

Saddle point in industrial context

Challenges:

- positive and negative eigenvalues $\Rightarrow$ classical iterative methods struggle
- modern applications require large systems $\Rightarrow$ direct solvers impractical

Active field of research: **robust** and **efficient** preconditioners

- (implicit) hypotheses: regular grid, M-matrices, ...
- theorems and convergence bounds

Industrial context: problems often outside the hypotheses
$\Rightarrow$ hope for the best.

Classical approach for 2×2 saddle point problems

Ideal block preconditioner:

$$\begin{pmatrix} A & B^T \\ -B & C \end{pmatrix} \Rightarrow \begin{pmatrix} A & \\ & S \end{pmatrix},$$

where $S = C + BA^{-1}B^T$ leads to optimal convergence.

Difficulty: efficient approximation of both A and S.

Stokes:

- A: discrete Laplacian $\Rightarrow$ AMG
- S: $BA^{-1}B^T \Rightarrow$ pressure mass matrix M

$\Rightarrow$ Ideal preconditioner

$$\begin{pmatrix} \tilde{A} & \\ & \tilde{M} \end{pmatrix} \tag{1}$$

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Navier-Stokes:

- A: convection-diffusion operator $\Rightarrow$ specific AMG
- S: $BA^{-1}B^T \Rightarrow$ no robust approximation with the Reynolds number

$\Rightarrow$ currently no robust preconditioner for incompressible Navier-Stokes.

Outline

PolyMAC: General principle

PolyMAC: Finite Volume (FV) **generalization** of the MAC scheme to polyhedral meshes. The main unknowns are then:

- normal component of the **velocity** at the **faces**
- **pressure** at the **cells**

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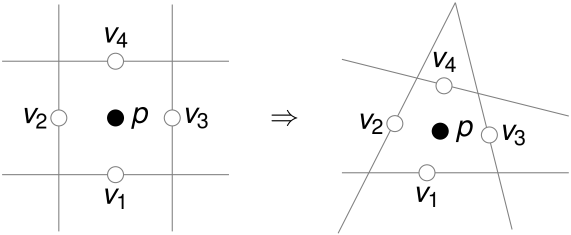
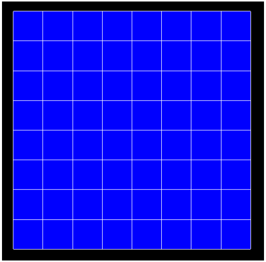
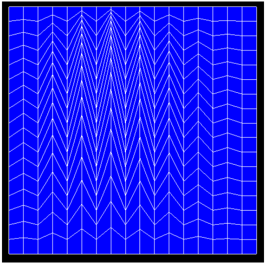


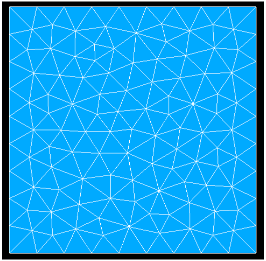
Meshes



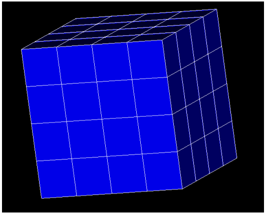
(a) Cartesian



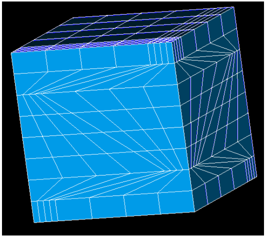
(b) Kershaw



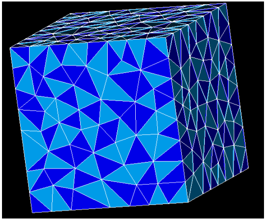
(c) Triangles



(d) Hexahedra



(e) Kershaw



(f) Tetrahedras

Towards industrial cases

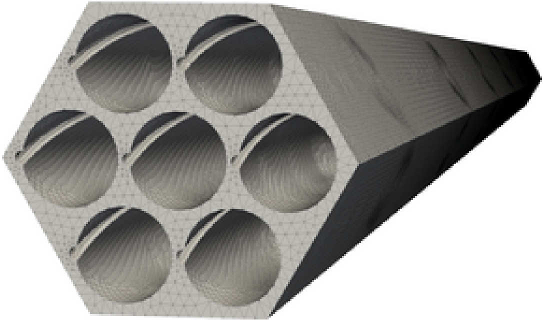


Figure 2: Sodium fast reactor `Assembly` mesh

- More complex than FVCA meshes
- Spacer => poor cells
- No satisfactory method without efficient linear solver

Outline

Incompressible Navier-Stokes equations

Find $\vec{u}$ and p such that

$$\begin{aligned} \partial_t \vec{u} + (\vec{u} \cdot \nabla)\vec{u} - \nu \Delta \vec{u} + \nabla p &= \vec{f} \quad \text{in } \Omega\,, \\ \nabla \cdot \vec{u} &= 0 \quad \text{in } \Omega\,, \end{aligned} \tag{1}$$

- $\vec{u}$: fluid velocity
- p: pressure
- $\nu > 0$: viscosity
- $\Omega \subset \mathbb{R}^d$: unit square in 2D (unit cube in 3D)

PolyMAC

Reformulation of Equation (1) with the vorticity:

$$\begin{aligned} \text{Momentum:} \quad & \partial_t \vec{u} + \nabla \cdot (\vec{u} \otimes \vec{u}) + \nu \nabla \times \vec{\omega} + \nabla p &= \vec{f}\,, \\ \text{Vorticity:} \quad & \nabla \times \vec{u} - \vec{\omega} &= 0\,, \\ \text{Divergence:} \quad & \nabla \cdot \vec{u} &= 0\,. \end{aligned} \quad (2)$$

- Mimetic finite volumes (Bonelle (2014), Beltman (2018))
- Stability (star-shaped mesh) and discrete conservation law
- Introduction of a dual mesh

PolyMAC: Linear System

Resulting linear system:

$$\begin{pmatrix} \frac{M^{(2)}}{\Delta t} + C([v]_F^t) & R & G \\ R^T & -\frac{1}{\nu} M^{(1)} & 0 \\ G^T & 0 & 0 \end{pmatrix} \begin{pmatrix} [v]_F^{t+\Delta t} \\ \nu\,[\omega]_A^{t+\Delta t} \\ [p]_{\overline{E}}^{t+\Delta t} \end{pmatrix} = \begin{pmatrix} \frac{M^{(2)}}{\Delta t}\mathbf{u}_f^t \\ 0 \\ 0 \end{pmatrix}, \tag{3}$$

- $C([v]_F^t)$: convection $\Rightarrow$ NS linearised at each time step
- Without convection, symmetric matrix
- **Saddle-point** system
- $M^{(2)}$: not diagonally dominant on deformed meshes

Outline

Prediction-correction method

Currently, the solving step is implemented as a correction-prediction method:

1. Prediction: Momentum equation on its own to find first approximation of $\vec{u}$
2. Correction: Poisson equation on pressure to enforce $\nabla \cdot \vec{u} = 0$

Linear system for the prediction:

$$\begin{pmatrix} \frac{M_U}{\Delta t} + F & R \\ R^T & -C \end{pmatrix} \begin{pmatrix} \mathbf{u}^\star \\ \nu\omega \end{pmatrix} = \begin{pmatrix} \frac{M_U}{\Delta t}\mathbf{u}^t - G\mathbf{p} \\ \mathbf{p} \end{pmatrix} \tag{4}$$

Solved by GMRES preconditioned by a block ILU method *i.e.* both the velocity block and the Schur complement are preconditioned by ILU.

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Linear system for the correction:

$$\begin{pmatrix} M_U & G \\ -G^T & 0 \end{pmatrix} \begin{pmatrix} \delta \mathbf{u} \\ \Delta t \delta \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ -G^T \mathbf{v}^\star \end{pmatrix} \tag{4}$$

Solved by a **direct solver** $\Rightarrow$ bottleneck of the approach.

Classical Approach for Saddle-Point problems

Block preconditioner for a Krylov subspace method:

$$\begin{pmatrix} M_U & \\ & S \end{pmatrix} \Rightarrow \begin{pmatrix} \tilde{M}_U & \\ & \tilde{S} \end{pmatrix} \quad (5)$$

- $\tilde{M}_U$: approximation of M_U
- $\tilde{S}$: approximation of $S = G^T M_U^{-1} G$ (typically: $G^T D_U^{-1} G$).

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Size	Iterations		
	Cartesian	Kershaw 2D	Assembly
1	8	66	NO
2	8	82	NO
3	8	93	NO
4	8	110	/
5	10	140	/
6	10	131	/

An algebraic transformation

Algebraic transformation of the system to give it a more suitable structure:

- Change of variables:

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} I & -D_U^{-1}G \\ & I \end{pmatrix} \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{p}} \end{pmatrix},$$

where D_U is the diagonal of M_U. The transformed system becomes:

$$\begin{pmatrix} M_U & (I - M_U D_U^{-1})G \\ -G^T & G^T D_U^{-1} G \end{pmatrix}$$

Advantages:

- More weight on the diagonal blocks
- Pressure block has a Laplacian-like structure: $\widehat{C} = G^T D_U^{-1} G$

Preconditioning the transformed system

Stokes-like system if reorder the transformed system ($\mathbf{u} \leftrightarrow \mathbf{p}$):

$$\begin{pmatrix} G^T D_U^{-1} G & -G^T \\ (I - M_U D_U^{-1})G & M_U \end{pmatrix}$$

State-of-the-art block preconditioner:

- $\widehat{C} = G^T D_U^{-1} G$: similar to a discrete Laplacian $\Rightarrow$ AMG preconditioner
- M_U: mass matrix *spectrally equivalent* to Schur complement $\Rightarrow 2M_U$

$$\begin{pmatrix} \tilde{\widehat{C}} & \\ & 2\tilde{M}_U \end{pmatrix}$$

Numerical Results

Size	Cartesian		Kershaw 2D		Assembly	
	Class.	Transf.	Class.	Transf.	Class.	Transf.
1	8	**5**	66	**35**	NO	**88**
2	8	**4**	82	**42**	NO	**49**
3	8	**4**	93	**49**	NO	**25**
4	8	**4**	110	**54**	NO	**28**
5	10	**4**	140	**56**	NO	**33**
6	10	**4**	131	**55**	NO	**33**

Conclusions

Overview:

- General FV scheme for Navier-Stokes equations
- Implemented for compressible multi-phasic flows
- Key issue identified: iterative solver
- Robust iterative solver for Correction step $\Rightarrow$ **full iterative process**

Perspectives:

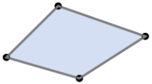
- Limitations due to the splitting approach
- Iterative solver for the full 3×3 system

Thank you! Any questions?

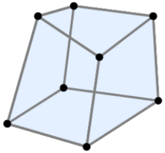
PolyMAC Control Volumes

(a) Primal Vertex s

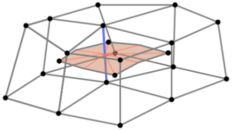
(b) Primal Edge a



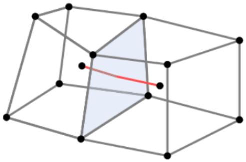
(c) Primal Face f



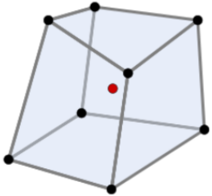
(d) Primal Cell e



(a) Dual Edge $\overline{a}$



(b) Dual Face $\overline{f}$



(c) Dual Cell $\overline{e}$

Mimetic methods for PolyMAC

Use of **mimetic** methods:

- Control volumes chosen so that differential operators have **exact** representation based on integral theorems such as

$$\int_e \nabla \cdot \vec{v} \; = \; \int_{\partial e} \vec{v} \cdot \vec{n}_{\partial e}$$

In terms of discrete unknowns, if the divergence is discretized at the cells and the velocity at the faces:

$$\left[\nabla \cdot \vec{v}\right]_e \; = \; \frac{1}{|e|} \sum_{f \vee e} |f| \, [v]_f \; = \; (D\,[v]_F)_e$$

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- **Approximation** introduced to link primal to dual mesh
  - the primal edges A to the dual edges $\overline{A}$:

$$[\omega]_{\overline{A}} \approx (M^{(1)} \, [\omega]_A)$$

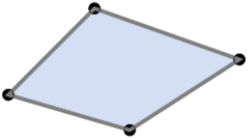
  - the primal faces F to the dual faces $\overline{F}$:

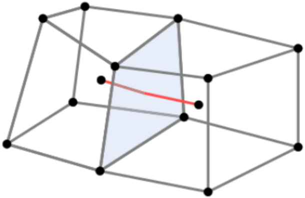
$$[v]_{\overline{F}} \approx (M^{(2)} \, [v]_F) \, .$$

Choice of Control Volumes

For PolyMAC I, the control volumes are

- Velocity at the faces f and momentum at the dual faces $\overline{f}$
  - Time derivative $\left[\partial_t \vec{v}\right]_{\overline{F}} = M^{(2)} \partial_t [v]_F$
  - Vorticity curl $[\nu \nabla \times \vec{\omega}]_{\overline{F}} = M^{(2)} R^F [\omega]_A$
  - Pressure gradient $[\nabla p]_{\overline{F}} = G [p]_{\overline{E}}$

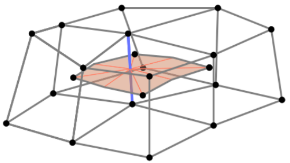




Choice of Control Volumes

For PolyMAC I, the control volumes are

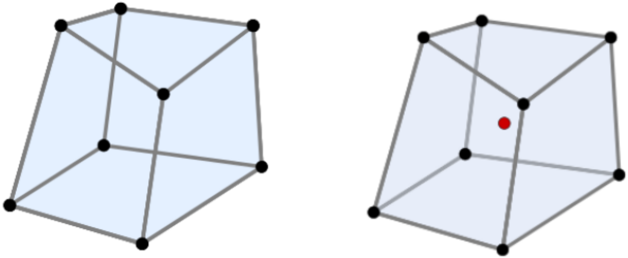
- Velocity at the faces f and momentum at the dual faces $\overline{f}$
- Vorticity at the edges a and its definition at the dual edges $\overline{a}$
  - Velocity curl $\left[\nabla \times \vec{v}\right]_{\overline{A}} = R^A M^{(2)} [v]_F$
  - Vorticity $[\omega]_{\overline{A}} = M^{(1)} [\omega]_A$



Choice of Control Volumes

For PolyMAC I, the control volumes are

- Velocity at the faces f and momentum at the dual faces $\overline{f}$
- Vorticity at the edges a and its definition at the dual edges $\overline{a}$
- Pressure at the dual cells $\overline{e}$ and continuity at the cells e
  - Divergence $\left[\nabla \cdot \vec{v}\right]_E = D\left[v\right]_F$



2D and 3D test problems

In 2D: rotation Navier-Stokes problem for a right-hand-side of the form:

$$\vec{u} = \begin{pmatrix} y \\ -x \end{pmatrix}, \tag{6}$$

with $p(x, y) = \frac{(x^2+y^2)}{2} - \frac{1}{3}$.

In 3D: Taylor-Green vortex problem (dependence in the viscosity).

(Extensively used in FVCA benchmarks)