General boundary conditions for the Boussinesq-Abbott model with varying topography

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Bordeaux

Objectives

Until november 2023: postdoc supported by **IMPT** → **ANR Bourgeons**

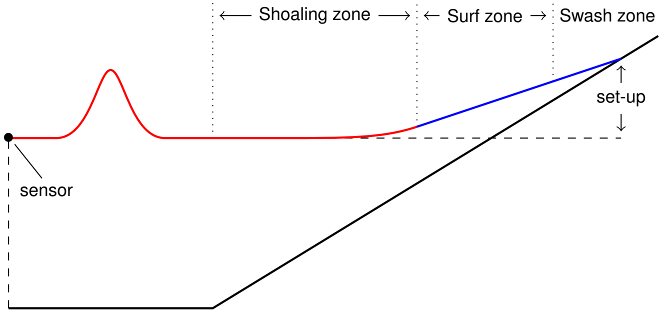
Supervision : David Lannes and Philippe Bonneton

Long term goal: study extreme waves in littoral area

Objectives

- Need accurate dispersive model: Boussinesq-type systems
- Boundary conditions are difficult to deal with

  Recently: Perfectly Matched Layer, source function method → costly



Boussinesq-Abbott model written in (ζ, q)-coordinates

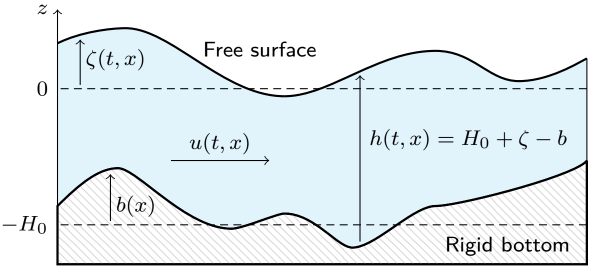
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 + h_b \mathcal{T}_b)\partial_t q + \partial_x f_{\mathrm{NSW}} = -gh\partial_x b \end{cases} \quad \text{in } (0, \ell)\,, \tag{BA}$$

under generating boundary conditions

$$\zeta(t, 0) = g_0(t), \qquad \zeta(t, \ell) = g_\ell(t),$$

with $h_b = H_0 - b$ (depth at rest) and

$$\mathcal{T}_b(\cdot) = -\frac{1}{3h_b}\partial_x\left(h_b^3 \partial_x \frac{(\cdot)}{h_b}\right) + \frac{(\cdot)}{2}\partial_x^2 b, \tag{1}$$



Problematic

How to account for boundary conditions? How to recover $q_{|_{x=0,\ell}}$?

- Hyperbolic case ($h_b \mathcal{T}_b \equiv 0$): Riemann invariants
- Dispersive case: need to invert $(1 + h_b \mathcal{T}_b)$ $\rightarrow$ requires knowledge on $\partial_t q_{|_{x=0,\ell}}$

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Lannes and Weynans 2020

Equivalent writing of Boussinesq-Abbott with flat bottom over $(0, \infty)$

- substitute $(1 + h_b \mathcal{T}_b)$ for nonlocal flux & dispersive boundary layer
- ODE on $q_{|x=0}$
- local existence and unicity
- 1st order scheme

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Outline of the talk

1. Reformulation over bounded domain
2. General boundary cond. & scheme
3. Varying bathymetry
4. Some perspectives

Reformulation of the model (flat bottom)

Flat bottom case ($b \equiv \mathrm{Cst}$): discharge eq. simplifies to

$$(1 - \kappa^2 \partial_{xx}^2)\partial_t q + \partial_x f_{\mathrm{NSW}}(\zeta, q) = 0 \qquad \text{in } (0, \ell)$$

Fix $0 \leq t \leq T$, then $y(x) = \partial_t q(t, x)$ satisfies an ODE of the form

$$\begin{cases} y - \kappa^2 y'' = \phi(x) \\ y(0) = \dot{q}_{|_0}, \quad y(\ell) = \dot{q}_{|_\ell} \end{cases}$$

Equivalently: $\quad y = y_h + y_b \quad$ with $\begin{cases} y_h - \kappa^2 y_h'' = 0 \\ y_h(0) = \dot{q}_{|_0}, \quad y_h(\ell) = \dot{q}_{|_\ell} \end{cases}$ and $\begin{cases} y_b - \kappa^2 y_b'' = \phi(x) \\ y_b(0) = y_b(\ell) = 0 \end{cases}$

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Define R^0 as the inverse of $(1-\kappa^2\partial_{xx}^2)$ with **homogeneous Dirichlet** conditions at $x = 0, \ell$

$$\Rightarrow \quad \partial_t q = \underbrace{-R^0\partial_x f_{\text{NSW}}}_{y_b} + \underbrace{\mathfrak{s}_{(0)}\dot{q}_{|_0} + \mathfrak{s}_{(\ell)}\dot{q}_{|_\ell}}_{y_h}$$

where $\begin{cases} (1-\kappa^2\partial_{xx}^2)\mathfrak{s}_{(0)} = 0 \\ \mathfrak{s}_{(0)}(0) = 1, \quad \mathfrak{s}_{(0)}(\ell) = 0 \end{cases}$ and $\begin{cases} (1-\kappa^2\partial_{xx}^2)\mathfrak{s}_{(\ell)} = 0 \\ \mathfrak{s}_{(\ell)}(0) = 0, \quad \mathfrak{s}_{(\ell)}(\ell) = 1 \end{cases}$. (2)

Reformulation of the model (flat bottom)

Note R^1 the inverse of $(1 - \kappa^2 \partial^2_{xx})$ with **homogeneous Neumann** conditions at $x = 0, \ell$

$$\boxed{\Rightarrow R^0 \partial_x = \partial_x R^1}$$

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Proposition 1 (D. Lannes, R.)

Assume a flat bottom $b \equiv 0$ *and let* $(\zeta, q)_{|_{t=0}} = (\zeta^{\text{in}}, q^{\text{in}})$. *The two assertions are equivalent:*

1. *The pair* (ζ, q) *satisfies* (BA) *with generating conditions* $\zeta(\cdot, 0) = g_0$ *and* $\zeta(\cdot, \ell) = g_\ell$
2. *The pair* (ζ, q) *satisfies*

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x (R^1 f_{\text{NSW}}) = \mathfrak{s}_{(0)} \dot{q}_{|_0} + \mathfrak{s}_{(\ell)} \dot{q}_{|_\ell} \end{cases} \quad \textit{in } (0, \ell), \tag{3}$$

with the trace ODEs

$$\begin{pmatrix} \mathfrak{s}'_{(0)}(0) & \mathfrak{s}'_{(\ell)}(0) \\ \mathfrak{s}'_{(0)}(\ell) & \mathfrak{s}'_{(\ell)}(\ell) \end{pmatrix} \begin{pmatrix} \dot{q}_{|_0} \\ \dot{q}_{|_\ell} \end{pmatrix} = \frac{1}{\kappa^2} \begin{pmatrix} (R^1 - \text{id})_{|_0} f_{\text{NSW}} \\ (R^1 - \text{id})_{|_\ell} f_{\text{NSW}} \end{pmatrix} - \begin{pmatrix} \ddot{g}_0 \\ \ddot{g}_\ell \end{pmatrix} \tag{4}$$

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Proof: ∂_x [discharge eq. (3)] $\Rightarrow$ $\underbrace{\partial_{xt}^2 q}_{-\partial_{tt}^2 \zeta} + \underbrace{(\partial_{xx} R^1 f_{\text{NSW}})}_{\frac{1}{\kappa^2}(R^1 - \text{id}) f_{\text{NSW}}} = \mathfrak{s}'_{(0)} \dot{q}_0 + \mathfrak{s}'_{(\ell)} \dot{q}_\ell$

More general boundary conditions

Possibility to enforce general boundary conditions

$$\xi_0^+(\zeta_{|_0}, q_{|_0})(t) = g_0(t), \qquad \xi_\ell^-(\zeta_{|_\ell}, q_{|_\ell})(t) = g_\ell(t). \tag{5}$$

For instance, $\xi^\pm$ given by q or Saint-Venant Riemann invariants

$$\mathcal{R}_\pm(U) = u \pm 2\sqrt{gh}$$

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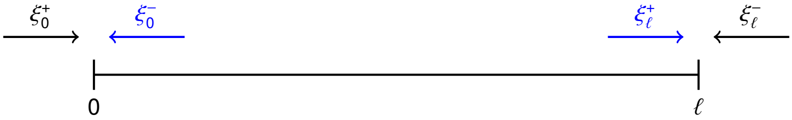
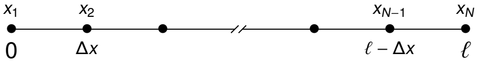
Adapt trace ODE in terms of missing data (outgoing information ξ_0^- and ξ_ℓ^+)

$$\begin{pmatrix} \mathfrak{s}'_{(0)}(0) & \mathfrak{s}'_{(\ell)}(0) \\ \mathfrak{s}'_{(0)}(\ell) & \mathfrak{s}'_{(\ell)}(\ell) \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} q(\xi_0^+, \xi_0^-) \\ q(\xi_\ell^+, \xi_\ell^-) \end{pmatrix} = \frac{1}{\kappa^2} \begin{pmatrix} (R^1 - \mathrm{id})_{|_0} f_{\mathrm{NSW}} \\ (R^1 - \mathrm{id})_{|_\ell} f_{\mathrm{NSW}} \end{pmatrix} - \frac{\mathrm{d}^2}{\mathrm{d}t^2} \begin{pmatrix} \zeta(\xi_0^+, \xi_0^-) \\ \zeta(\xi_\ell^+, \xi_\ell^-) \end{pmatrix}$$



Numerical schemes for the reformulated system

Discretize $(0, \ell)$ as follows:



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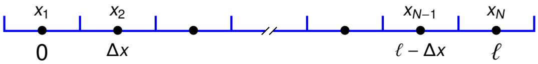

Note $U_i^n = (\zeta_i^n, q_i^n)^T$ the approximation of $\dfrac{1}{\Delta x} \displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} \begin{pmatrix} \zeta \\ q \end{pmatrix} (t^n, s)\, ds$.

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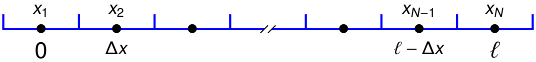

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Time stepping procedure

Step 1: Define $\underline{R}^1 f_{\mathrm{NSW}}^n$ as the vector $v \in \mathbb{R}^N$ satisfying

$$\begin{cases} v_i - \kappa^2 \dfrac{v_{i+1} - 2v_i + v_{i-1}}{\Delta x^2} = f_{\mathrm{NSW}}(U_i^n) \;\text{ for }\; 2 \le i \le N-1 \\ \dfrac{v_2 - v_1}{\Delta x} = \dfrac{v_N - v_{N-1}}{\Delta x} = 0 \end{cases} .$$

Similar definition for boundary layer functions $\underline{\mathfrak{s}}_{(0)}$ and $\underline{\mathfrak{s}}_{(\ell)}$.

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Time stepping procedure

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$$\begin{cases} \zeta_1^{n+1} = \zeta(g_0^{n+1}, (\xi_0^-)^{n+1}) \\ q_1^{n+1} = q(g_0^{n+1}, (\xi_0^-)^{n+1}) \end{cases} , \qquad \begin{cases} \zeta_N^{n+1} = \zeta((\xi_\ell^+)^{n+1}), g_\ell^{n+1}) \\ q_N^{n+1} = q((\xi_\ell^+)^{n+1}, g_\ell^{n+1}) \end{cases}$$

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Step 4: For $2 \leq i \leq N-1$, finite volumes update with Lax-Friedrichs numerical flux

$$\begin{cases} \dfrac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t} + \dfrac{1}{\Delta x}\left(q_{i+1/2}^n - q_{i-1/2}^n\right) = 0 \\ \dfrac{q_i^{n+1} - q_i^n}{\Delta t} + \dfrac{1}{\Delta x}\left((\underline{R}^1 f_{\mathrm{NSW}}^n)_{i+1/2} - (\underline{R}^1 f_{\mathrm{NSW}}^n)_{i-1/2}\right) = (\mathfrak{s}_{(0)})_i \delta_t q_1^n + (\mathfrak{s}_{(\ell)})_i \delta_t q_N^n \end{cases}.$$

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Extension to 2nd order: Mac-Cormack method (prediction-correction)

Case of a varying topography ($b \not\equiv \text{Cst}$)

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Note R_b^0 the inverse of $(1 + h_b \mathcal{T}_b)$ with **homogeneous Dirichlet cond.** $R_b^0 \partial_x = \partial_x \ldots ?$

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Lemma 1 (generalization of $R^0\partial_x = \partial_x R^1$)

We can construct a nonlocal operator R_b^1 such that

$$R_b^0\partial_x(\cdot) = \left(\partial_x + \beta + \frac{\partial_x\alpha}{\alpha}\right)\left[\frac{h_b^2}{\alpha}R_b^1\left(\frac{(\cdot)}{h_b^2}\right)\right] - R_b^0\big((\cdot)\beta\big) \quad \textit{with} \quad \begin{cases} h_b = H_0 - b \\ \alpha = 1 + \frac{1}{4}(\partial_x b)^2 \\ \beta = \frac{3}{2}h_b^{-1}\partial_x b \end{cases}$$

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Definition 1 (Nonlocal flux and source terms)

$$\mathfrak{f} = \frac{h_b^2}{\alpha}R_b^1\left(\frac{f_{\text{NSW}}}{h_b^2}\right), \qquad \mathfrak{B} = R_b^0\Big(-gh\partial_x b + \beta f_{\text{NSW}}\Big) - \left(\beta + \frac{\partial_x\alpha}{\alpha}\right)\mathfrak{f}$$

Proposition 2 (D. Lannes, R.)

Let $(\zeta, q)_{|t=0} = (\zeta^{\text{in}}, q^{\text{in}})$. *The two assertions are equivalent:*

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2. *The pair* (ζ, q) *satisfies*

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \mathfrak{f}(\zeta, q) = \mathfrak{B}(\zeta, q) + \mathfrak{s}_{(b,0)} \dot{q}_{|_0} + \mathfrak{s}_{(b,\ell)} \dot{q}_{|_\ell} \end{cases} \quad \text{in } (0, \ell) \tag{6}$$

and the trace equations

$$\begin{pmatrix} \mathfrak{s}'_{(b,0)}(0) & \mathfrak{s}'_{(b,\ell)}(0) \\ \mathfrak{s}'_{(b,0)}(\ell) & \mathfrak{s}'_{(b,\ell)}(\ell) \end{pmatrix} \begin{pmatrix} \dot{q}_{|_0} \\ \dot{q}_{|_\ell} \end{pmatrix} + V_{boundary}(g_0, q_{|_0}, g_\ell, q_{|_\ell}) = V_{interior}[\zeta, q] - \begin{pmatrix} \ddot{g}_0 \\ \ddot{g}_\ell \end{pmatrix}. \tag{7}$$

where $V_{boundary}$, $V_{interior}$ *are known.*

Numerical schemes

- Standard finite differences for nonlocal terms and trace ODEs
- Finite volumes for interior equations (Lax-Friedrichs or MacCormack)

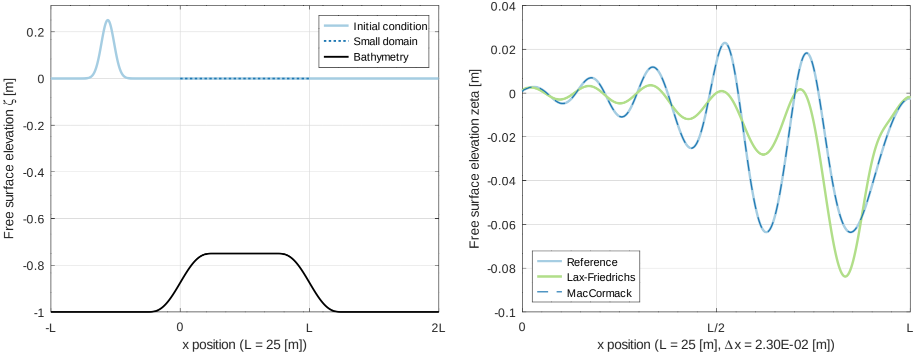

Figure: Gaussian over bump (left: initial time, right: t = 15 [s])

Question: starting from a wrong initial condition, can we recover the reference solution by enforcing appropriate boundary conditions?

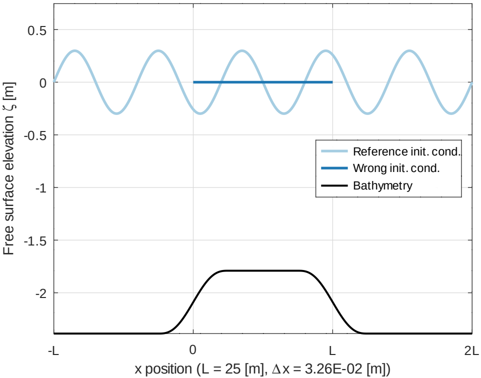

Figure: Sine over bump

Numerical schemes

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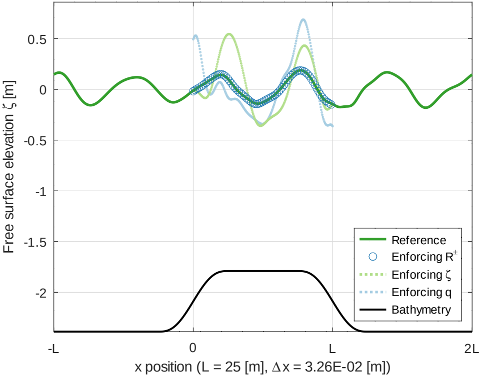
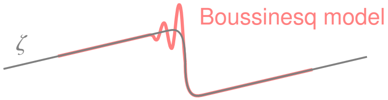
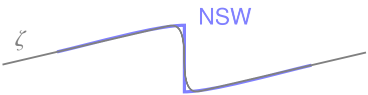


Figure: Sine over bump

Perspectives: coupling with the shallow water model

Motivation: wave breaking with dispersive models → non physical oscillations.





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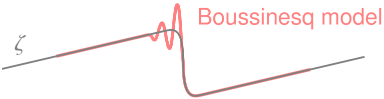
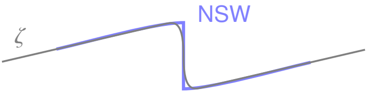
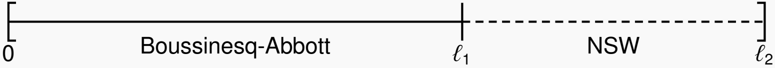




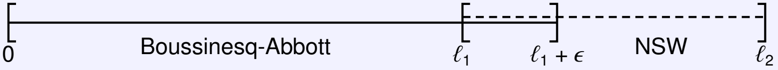
$$
\begin{cases}
\partial_t \zeta_L + \partial_x q_L = 0 \\
\partial_t q_L + \partial_x \left(\frac{h_b^2}{\alpha_b} R_b^1 \left(\frac{1}{h_b^2} f_{NSW} \right) \right) = \mathfrak{S}(\zeta_L, q_L) + \mathfrak{s}_{(0)} \dot{q}_{L|_{x=0}} + \mathfrak{s}_{(\ell_1)} \dot{q}_{L|_{x=\ell_1}} & \text{in } (0, \ell_1) \\
\partial_t \zeta_R + \partial_x q_R = 0 \\
\partial_t q_R + \partial_x f_{NSW}(U_R) = -g h_R \partial_x b & \text{in } (\ell_1, \ell_2)
\end{cases}
$$

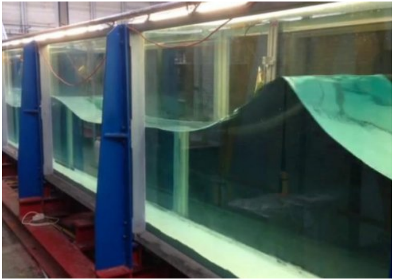
Coupling conditions: $\xi^+_{\ell_1}(U_{R|_{\ell_1}}) = \xi^+_{\ell_1}(U_{L|_{\ell_1}}), \quad \xi^-_{\ell_1}(U_{L|_{\ell_1}}) = \xi^-_{\ell_1}(U_{R|_{\ell_1}})$



Preliminary observations and ideas

- Artifacts near coupling interface → improved with a spatial overlapping...
- ... but difficult to interpret at continuous level





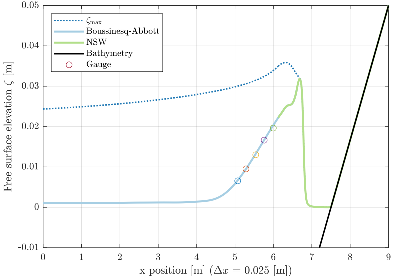


Figure: Canal à houle du LEGI (Grenoble) ; comparaison données expérimentales/simulation

Perspectives: extreme waves statistics

- Random wave generation from boundary conditions (with Philippe Bonneton)
- Implement method in DGFEM code (C++, with Fabien Marche & Lisl Weynans)

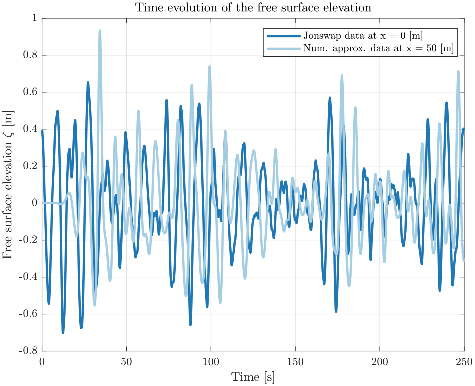


Figure: Random signal for the elevation

Conclusion

- Boussinesq-Abbott model is accurate, but boundary conditions are challenging
- Reformulation strategy allows to recover missing data
- Extension to varying bathymetries + general boundary conditions
- Efficient 1st and 2nd order schemes
- Numerical experiments (asymptotic stability, coupling with shallow water eq.)

Perspective: high order DGFEM code, study of extreme waves

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Thank you for your attention!