Neumann boundary controllability of a semilinear wave equation

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Framework. Let $\Omega = (0,1)$ and $T > 0$. We denote

$$Q_T := \Omega \times (0,T), \qquad H^1_{(0)}(\Omega) := \{z \in H^1(\Omega);\ z(0) = 0\}$$

We consider

$$\begin{cases} \partial_{tt} y - \partial_{xx} y + f(y) = 0, & Q_T, \\ y(0,\cdot) = 0,\ \partial_x y(1,\cdot) = v, & (0,T), \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1), & \Omega, \end{cases} \qquad (\star)$$

where $(u_0, u_1) \in H^1_{(0)}(\Omega) \times L^2(\Omega)$, $v \in L^2(0,T)$ and $f \in \mathcal{C}^1(\mathbb{R})$ is a non-linear function.

Framework and objective

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Exact controllability problem.

Given $T > 0$ and $(u_0,u_1),(z_0,z_1) \in H^1_{(0)}(\Omega) \times L^2(\Omega)$, find a control $v \in L^2(0,T)$ and a solution $y \in \mathcal{C}^0([0,T]; H^1_{(0)}(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$ of $(\star)$ such that

$$(y(\cdot,T), \partial_t y(\cdot,T)) = (z_0,z_1).$$

First main controllability result

Theorem 1 - Claret 2024

Let $T > 2$. Assume that $f \in \mathcal{C}^1(\mathbb{R})$ satisfies

$$\limsup_{|r|\to+\infty} \frac{|f(r)|}{r \ln^2 |r|} < \beta_0, \tag{H1}$$

with $\beta_0 > 0$ small enough. Then, system $(\star)$ is exactly controllable in $H^1_{(0)}(\Omega) \times L^2(\Omega)$.

2. for the cost $\mathcal{J}(y, v) = \|y\|^2_{L^2(Q_T)} + \|v\|^2_{L^2(0,T)}$.

1. Enrique Zuazua - Exact controllability for semilinear wave equations in one space dimension, In Annales de l'Institut Henri Poincaré, 1993

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$\rightarrow$ based on Schauder fixed point argument as in [Zuazua, 93][1] :

Let $K : L^\infty(Q_T) \to L^\infty(Q_T)$, $K(z) := y$ where (y, v) is the optimal[2] pair solution of

$$\begin{cases} \partial_{tt} y - \partial_{xx} y + \widehat{f}(z) y = -f(0), & Q_T, \\ y(0,t) = 0, \ \partial_x y(1,t) = v, & (0,T), \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1), & \Omega, \\ (y(\cdot,T), \partial_t y(\cdot,T)) = (z_0, z_1), & \Omega, \end{cases} \qquad \widehat{f}(r) = \begin{cases} \dfrac{f(r) - f(0)}{r} & r \neq 0 \\ f'(0) & r = 0. \end{cases} \tag{SL}$$

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If $\beta_0 > 0$ small enough, the operator K admits a fixed point : $\exists C \subset L^\infty(Q_T)$ s.t.

$$K(C) \subset C, \qquad K : C \to C \text{ is continuous}, \qquad \overline{K(C)} \text{ is compact .}$$

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Theorem 2 - Claret 2024

Let $T > 2$ and $A \in L^\infty(Q_T)$. Then, there exists $C = C(\Omega, T) > 0$ such that

$$\|(\varphi(\cdot,0), \partial_t\varphi(\cdot,0))\|_{L^2(\Omega)\times H^{-1}_{(0)}(\Omega)} \leq Ce^{C\sqrt{\|A\|_\infty}} \left(\|\varphi(1,\cdot)\|_{L^2(0,T)} + \|\partial_{tt}\varphi - \partial_{xx}\varphi + A\varphi\|_{L^2(Q_T)}\right), \quad (1)$$

for all $\varphi \in \left\{\varphi \in \mathcal{C}^0\left[0,T\right]; L^2(\Omega)\right) \cap \mathcal{C}^1\left([0,T]; H^{-1}_{(0)}(\Omega)\right);\ \partial_{tt}\varphi - \partial_{xx}\varphi + A\varphi \in L^2(Q_T)\right\}$.

Observability inequality implies controllability result in linear case

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for all $\varphi \in \left\{\varphi \in \mathcal{C}^0\left[0,T]; L^2(\Omega)\right) \cap \mathcal{C}^1\left([0,T]; H^{-1}_{(0)}(\Omega)\right);\ \partial_{tt}\varphi - \partial_{xx}\varphi + A\varphi \in L^2(Q_T)\right\}$.

Lake of convergence of Fixed point algorithm

The operator K is not a priori contracting in general : The Picard iterate $(y_k)_{k\in\mathbb{N}}$ given by

$$y_0 \in L^\infty(Q_T), \qquad y_{k+1} = K(y_k), \quad \forall k \in \mathbb{N} \tag{FP}$$

is bounded but not convergent.

Second main controllability result

Used a least-squares approach as in [Münch and Trélat, 22][3]

Theorem 3 - Claret 2024

Let $T > 2$. Assume that $f \in \mathcal{C}^1(\mathbb{R})$ such that f' is α-Hölder continuous for some $\alpha \in (0,1]$ and satisfies

$$\limsup_{|r|\to+\infty} \frac{|f'(r)|}{\ln^2|r|} < \beta_1, \tag{H2}$$

with β_1 small enough. Then, there exists a sequence $(y_k, v_k)_{k\in\mathbb{N}}$ which strongly converges to a state-control pair $(y, v) \in \mathcal{A}$ of $(\star)$. The convergence is at least linear, then at least of order $1+\alpha$ after a finite number of iterations.

3. Arnaud Münch, Emmanuel Trélat - Constructive exact control of semilinear 1D wave equations by a least-squares approach, SIAM J. Control and Optimization, 2022

Sketch of the proof

We consider the Hilbert space

$$\mathcal{H} := \Big\{(y,v) \in L^2(Q_T) \times L^2(0,T);\ y \in \mathcal{C}^0([0,T]; H^1_{(0)}(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega)),\\ y(0,\cdot) = 0,\ \partial_x y(1,\cdot) = v,\ \partial_{tt} y - \partial_{xx} y \in L^2(Q_T)\Big\}$$

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$$y(0,\cdot) = 0,\ \partial_x y(1,\cdot) = v,\ \partial_{tt} y - \partial_{xx} y \in L^2(Q_T)\Big\}$$

and the two closed subspaces

$$\mathcal{A} := \Big\{(y,v) \in \mathcal{H};\ \big(y(\cdot,0), \partial_t y(\cdot,0)\big) = (u_0, u_1),\ \big(y(\cdot,T), \partial_t y(\cdot,T)\big) = (z_0, z_1)\Big\}$$
$$\mathcal{A}_0 := \Big\{(y,v) \in \mathcal{H};\ \big(y(\cdot,0), \partial_t y(\cdot,0)\big) = (0,0),\ \big(y(\cdot,T), \partial_t y(\cdot,T)\big) = (0,0)\Big\}$$

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Non convex minimization problem

$$\min_{(y,v)\in\mathcal{A}} E(y,v), \qquad E(y,v) := \frac{1}{2}\|\partial_{tt} y - \partial_{xx} y + f(y)\|^2_{L^2(Q_T)}$$

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Goal : Compute $(y_k, v_k)_{k\in\mathbb{N}} \subset \mathcal{A}$ such that $(y_k, v_k) \to (y,v)$ where $E(y,v) = 0$.

Sketch of the proof

Proposition

Let $T > 2$. For any $(y, v) \in \mathcal{A}$, there exists a constant $C = C(\Omega, T) > 0$ such that

$$\sqrt{E(y,v)} \leq C e^{C\sqrt{\|f'(y)\|_\infty}} \|E'(y,v)\|_{\mathcal{A}_0'}$$

where $\mathcal{A}_0'$ is the topological dual of $\mathcal{A}_0$.

Sketch of the proof

For any $(y, v) \in \mathcal{A}$, we denote by $(Y, V) \in \mathcal{A}_0$ the optimal state-control pair solution of

$$\begin{cases} \partial_{tt} Y - \partial_{xx} Y + f'(y) Y = \partial_{tt} y - \partial_{xx} y + f(y), & Q_T, \\ Y(0, \cdot) = 0, \ \partial_x Y(1, \cdot) = V, & (0, T), \\ \big(Y(\cdot, 0), \partial_t Y(\cdot, 0)\big) = (0, 0), & \Omega, \\ \big(Y(\cdot, T), \partial_t Y(\cdot, T)\big) = (0, 0), & \Omega, \end{cases}$$

for the cost $\mathcal{J}(y, v) = \|y\|^2_{L^2(Q_T)} + \|v\|^2_{L^2(0,T)}$.

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for the cost $\mathcal{J}(y,v) = \|y\|^2_{L^2(Q_T)} + \|v\|^2_{L^2(0,T)}$.

$\rightarrow$ We have

$$E'(y,v) \cdot (Y,V) = 2E(y,v) \geq 0.$$

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→ We have

$$\boxed{E'(y,v)\cdot(Y,V) = 2E(y,v) \geq 0.}$$

Least squares algorithm

We consider the sequence $(y_k, v_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ defined by

$$\begin{cases} (y_0, v_0) \in \mathcal{A} \\ (y_{k+1}, v_{k+1}) = (y_k, v_k) - \lambda_k (Y_k, V_k) \\ \lambda_k = \operatorname{argmin}_{\lambda\in[0,1]} E\left((y_k, v_k) - \lambda (Y_k, V_k)\right). \end{cases} \tag{LS}$$

Numerical illustrations of the least squares algorithm convergence

- We consider $\Omega = (0,1)$ and $T = 3$.
- We take $u_0 = 20\,(\cos(\pi x) - 1)$, $u_1 = 0$ and we control to zero.
- For any $c_f \in \mathbb{R}$, we consider the non-linear function

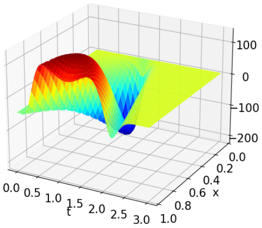
$$f(r) = c_f\, r \ln^2(1 + |r|), \qquad \forall r \in \mathbb{R}.$$

- We initialize the algorithm with (y_0, v_0) the linear state-control pair (with $f = 0$).
- The solutions of the wave equations are approximated using space-time finite element approach.
- We denote by

$$k^\star := \min_{k \geq 0} \left\{ \sqrt{2E(y_k, v_k)} \leq 10^{-5} \right\}.$$

Numerical simulations of the least-squares algorithm convergence

$u_0 = 20(\cos(\pi x) - 1)$, $u_1 = 0$, $f(r) = c_f\, r \ln^2(1 + |r|)$.



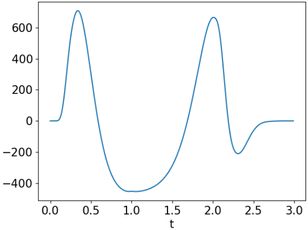


Figure – $c_f = -1$ - Representation of $y_{k^\star}$ (in left) and $v_{k^\star}$ (in right).

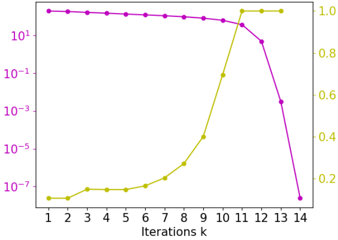


Figure – $c_f = -1$ - Evolution of $\sqrt{2E(y_k, v_k)}$ ( ) and λ_k ( ) w.r.t. k.

Numerical illustrations of the least squares algorithm convergence

$u_0 = 20\,(\cos(\pi x) - 1),\ u_1 = 0,\ f(r) = c_f\, r\, \ln^2(1 + |r|).$

c_f	$\sqrt{2E(y_{k^\star}, v_{k^\star})}$	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(0,T)}$	$\frac{\|y_{k^\star}-y_{k^\star-1}\|_{L^2}}{\|y_{k^\star-1}\|_{L^2}}$	$\frac{\|v_{k^\star}-v_{k^\star-1}\|_{L^2}}{\|v_{k^\star-1}\|_{L^2}}$	$k^\star$
10	5.461×10^{-9}	3.635×10^{1}	1.547×10^{3}	4.805×10^{-7}	6.015×10^{-7}	27
5	1.726×10^{-9}	2.478×10^{1}	3.449×10^{2}	9.589×10^{-8}	1.654×10^{-8}	9
2	5.754×10^{-10}	2.025×10^{1}	8.554×10^{1}	4.392×10^{-8}	2.470×10^{-7}	7
1	2.512×10^{-8}	2.099×10^{1}	5.895×10^{1}	5.857×10^{-6}	2.399×10^{-5}	5
-0.5	3.656×10^{-6}	4.195×10^{1}	1.355×10^{2}	9.689×10^{-5}	2.928×10^{-4}	5
-1	2.409×10^{-8}	1.145×10^{2}	6.003×10^{2}	2.826×10^{-7}	6.473×10^{-7}	13
-1.5	3.314×10^{-8}	3.332×10^{2}	2.541×10^{3}	4.538×10^{-8}	1.635×10^{-7}	40
-2	1.217×10^{-9}	9.982×10^{2}	1.110×10^{4}	5.914×10^{-10}	1.408×10^{-9}	143

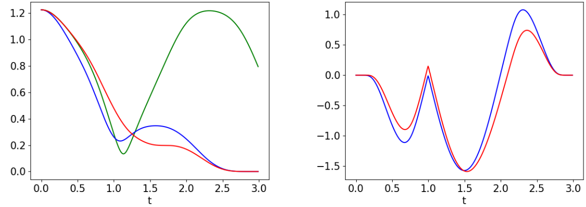
Table – Norm of $y_{k^\star}$ and $v_{k^\star}$ w.r.t. c_f.

Numerical simulations of the least squares algorithm convergence

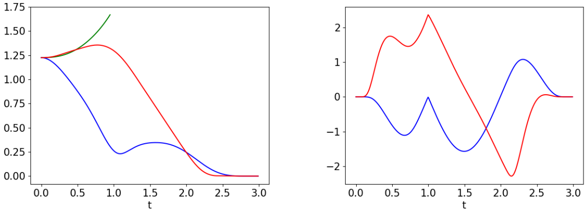
$u_0 = cos(\pi x) - 1$, $u_1 = 0$

Right : Representation of the uncontrolled solution $\|y^\star(\cdot,t)\|_{L^2(\Omega)}$ (green line), linear solution $\|y_0(\cdot,t)\|_{L^2(\Omega)}$ (blue line) and semilinear solution $\|y_{k^\star}(\cdot,t)\|_{L^2(\Omega)}$ (red line). **Left** : Representation of linear control v_0 (blue line) and semilinear control $v_{k^\star}$ (red line) for $c_f = -1$ and $c_f = -4$.

$c_f = -1$



$c_f = -4$



Conclusion

- We obtain an observability inequality with an explicit expression of the constant as an exponential function of the potential
- Global controllability result with an optimal assumption on f
  $\rightarrow$ extend to Neumann boundary control the results in [Zuazua, 93]
- Under stronger assumption on f, we get a constructive proof of controllability
  $\rightarrow$ extend to boundary case the least squares method introduced in [Münch and Trélat, 22]

Perspectives

- Prove a contraction property of the fixed-point operator K
  $\rightarrow$ by introducing a Carleman parameter [Bhandary and al, 2023][4], [Claret and al, 2024][5]
- Multidimensional case ?
- Application of the least squares method to inverse problem ?

4. Kuntal Bhandary, Jérôme Lemoine and Arnaud Münch - Exact boundary controllability of 1d semilinear wave equations through a constructive approach, Mathematics of Control, Signals, and Systems, 2023

5. Sue Claret, Jerôme Lemoine and Arnaud Münch - On the exact boundary controllability of semilinear wave equations, SIAM J. Control and Optimization, to appear

Thank you for your attention