Magnetic skyrmions confined in a bounded domain

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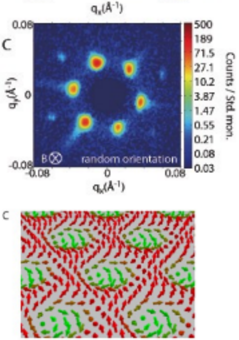
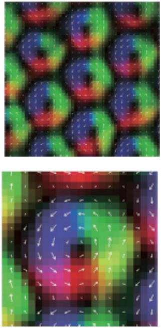
Magnetic skyrmions

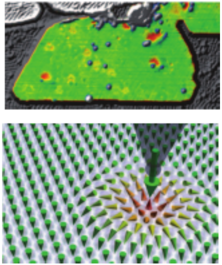
Skyrmions are particule-like topological singularities in non-centrosymmetric ferromagnets first predicted (Bogdanov-Hubert, Bogdanov-Kudinov-Yablonskii), then observed:



Mühlbauer et al.
Science 323, 2009



Yu et al.
Nature 465, 2010



Romming et al.
Science 341, 2013

Topological degree

A H^1 map $m : \mathbb{R}^2 \to \mathbb{S}^2$ **constant at infinity** can be identified with a map $\mathbb{S}^2 \to \mathbb{S}^2$; it carries a topological degree

$$\deg(m) = \frac{1}{4\pi} \int_{\mathbb{R}^2} m \cdot \partial_1 m \times \partial_2 m \in \mathbb{Z}$$

(number of times m covers $\mathbb{S}^2$ counted with orientation)

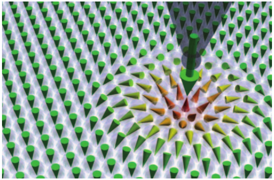


Figure: Degree 1 configuration

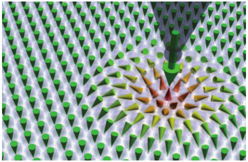
Micromagnetic energy and DMI interaction

Given $m = (m', m_3) : \Omega \subset \mathbb{R}^2 \to \mathbb{S}^2$ magnetization,

$$E(m) = \underbrace{d^2 \int_\Omega |\nabla m|^2}_{\text{Exchange energy}} + \underbrace{2\kappa \int_\Omega m' \cdot \nabla m_3}_{\text{DMI}} + \underbrace{\int_\Omega \phi(m)}_{\text{Anisotropy + Zeeman}} + \underbrace{\int_{\mathbb{R}^3} |H|^2}_{\text{Stray field}}$$

The Dzyaloshinskii-Moriya interaction (DMI), or antisymmetric exchange energy, promotes rotation of the magnetization vector:

$$\int_\Omega m' \cdot \nabla m_3 = \int_{-1}^{1} \left(\int_{\partial\{m_3 > t\}} -m' \cdot n_{\mathrm{ext}} \, \mathrm{d}s \right) \mathrm{d}t$$



Existence of skyrmions in infinite thin films

Reduced 2D magnetic energy: $m : \mathbb{R}^2 \to \mathbb{S}^2$,

$$E(m) = \int_{\mathbb{R}^2} |\nabla m|^2 + 2\kappa \int_{\mathbb{R}^2} m' \cdot \nabla m_3 + h \int_{\mathbb{R}^2} |m + e_3|^2$$

If $h > 0$ is fixed, and $\kappa > 0$ is small enough then E has a minimizer among degree 1 configurations. [Melcher, 2014]

Similar results with the stray field energy [Bernand-Mantel, Muratov, Simon, 2021]

Analogue to the Skyrme functional from nuclear physics [Esteban, '86, '90]

Topological lower bound

If $m : \mathbb{R}^2 \to \mathbb{S}^2$ is a degree 1 map constant at infinity, then

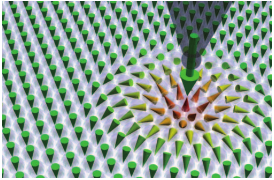
$$\int_{\mathbb{R}^2} |\nabla m|^2 \geq 8\pi$$

Equality is achieved by the harmonic maps [Belavin-Polyakov, '75]

$$R\Phi(\frac{x - x_0}{r}), \quad R \in SO(3), r > 0, x_0 \in \mathbb{R}^2,$$

where $\Phi : \mathbb{R}^2 \to \mathbb{S}^2$ is the inverse of the stereographic projection

$$\Phi(x) = \frac{1}{1 + |x|^2}(2x, 1 - |x|^2)$$

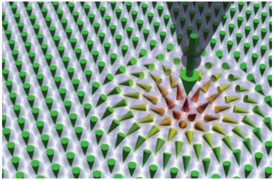


Upper energy bound

There exists an admissible m such that

$$\int_{\mathbb{R}^2} |\nabla m|^2 + 2\kappa \int_{\mathbb{R}^2} m' \cdot \nabla m_3 + h \int_{\mathbb{R}^2} |m + e_3|^2 < 8\pi$$

We use a well-chosen harmonic map $R\Phi(\frac{x-x_0}{r})$ with a correction at the tail so that the anisotropy remains finite.



Existence of skyrmions in confined geometry

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. We consider

$$\min \left\{ \int_\Omega |\nabla m|^2 + 2\kappa \int_\Omega m' \cdot \nabla m_3 \ : \ m = -e_3 \text{ on } \partial\Omega, \ \deg(m) = 1 \right\}$$

The boundary condition can be achieved by :

- a boundary penalty induced by the stray field in a thin film limit :
[Kohn - Slastikov, '05] [Ignat - Kurzke, '22]
[Di Fratta - Muratov - Slastikov, '24]

- patterning the substrate with strongly anchoring material :
[Ohara-Zhang-Chen-Wei-Ma-Xia-Zhou, '21]

Proposition

If $\kappa \ll 1$, there exists a minimizer.

Asymptotics of minimizers when $\kappa \to 0$

By the energy bounds, for a family of minimizers m_κ,

$$\int_\Omega |\nabla m_\kappa|^2 \to 8\pi \quad \text{as } \kappa \to 0.$$

Hence, we expect m_κ to be close to a harmonic map in some sense.

Proposition

There exist $R_\kappa \in SO(3)$, $r_\kappa > 0$, $x_\kappa \in \mathbb{R}^2$ s.t.

$$\left\| \nabla \left(m_\kappa - R_\kappa \Phi\left(\frac{x - x_\kappa}{r_\kappa}\right)\right) \right\|_{L^2} \to 0$$

Quantitative stability estimate for harmonic maps

Theorem (Bernand-Mantel, Muratov, Simon, 2021)

For all map $m : \mathbb{R}^2 \to \mathbb{S}^2$ of degree 1 and constant at infinity, we have

$$\int_{\mathbb{R}^2} |\nabla m|^2 - 8\pi \geq C \inf_{\psi \text{ harmonic}} \int_{\mathbb{R}^2} |\nabla m - \nabla \psi|^2$$

for some $C > 0$.

See also [Topping, 2020], [Hirsch, Zemas, 2021], [Deng, Sun, Wei, 2021], [Rupflin, 2023]

Next order energy asymptotics

Let (m_κ) be a family of minimizers for

$$E_\kappa(m) = \int_{\mathbb{R}^2} |\nabla m|^2 + 2\kappa m' \cdot \nabla m_3, \quad m = -e_3 \text{ on } \mathbb{R}^2 \backslash \Omega, \ \deg(m) = 1$$

By the stability estimate and energy bounds, when $\kappa \to 0$,

$$m_\kappa = R_\kappa \Phi\left(\frac{x - x_\kappa}{r_\kappa}\right) + o(1) \qquad \text{and} \qquad E_\kappa(m_\kappa) \simeq 8\pi - C\kappa^2$$

Energy asymptotics

Theorem (M-Muratov-Slastikov-Simon)

For family of minimizers (m_κ) *with* $m_\kappa = -e_3$ *on* $\mathbb{R}^2 \setminus \Omega$, *we have*

$$m_\kappa = R_\kappa \Phi\left(\frac{x - x_\kappa}{\kappa r_\kappa}\right) + \kappa u_\kappa + cste_\kappa$$

with, up to a subsequence

$$(R_\kappa, x_\kappa, r_\kappa, u_\kappa) \to (R, x_0, r, u) \in SO(3) \times \Omega \times \mathbb{R}^+ \times H^1_{\mathrm{loc}}$$

Moreover (R, x, r, u) *minimizes the limiting energy :*

$$r^2 \int_{\mathbb{R}^2} |\nabla u|^2 + 2r \int_{\mathbb{R}^2} (R\Phi)' \cdot \nabla(R\Phi \cdot e_3), \quad u = 2\frac{x - x_0}{|x - x_0|^2} \text{ on } \Omega^c$$

The last expresion is the Γ-limit of $\frac{E_\kappa(m_\kappa) - 8\pi}{\kappa^2}$

Renormalized energy

Minimizing in each variable of the limiting energy

$$r^2 \int_{\mathbb{R}^2} |\nabla u|^2 + 2r \int_{\mathbb{R}^2} (R\Phi)' \cdot \nabla (R\Phi \cdot e_3), \quad u = 2\frac{x - x_0}{|x - x_0|^2} \text{ on } \mathbb{R}^2 \setminus \Omega$$

gives successively :

- $R = \mathrm{id}$,
- $x_0 \in \operatorname{argmin} T(x_0)$ with

$$T(x_0) = \min \left\{ \int_{\mathbb{R}^2} |\nabla u|^2 \ : \ u = 2\frac{x - x_0}{|x - x_0|^2} \text{ on } \mathbb{R}^2 \setminus \Omega \right\},$$

- $r = 4\pi / T(x_0)$,

Explicit geometries

Proposition

Let $\Omega = B(0,1)$, *then*

$$T(x_0) = \frac{16\pi}{(1-|x_0|^2)^2}$$

which is minimal at the center $x_0 = 0$.

Proposition

Let $\Omega = \mathbb{R} \times (-\frac{1}{2}, \frac{1}{2})$, *then*

$$T(x_0, y_0) = \frac{4\pi^3}{\cos^2(\pi y_0)}$$

which is minimal when $y_0 = 0$.

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Merci pour votre attention!