Disentangling entropy and suboptimality in Entropic optimal transport

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The regularized optimal transport

Let $\rho \in \mathcal{P}_{2,ac}(\mathbb{R}^k)$ be a continuous measure with finite variance, define

$$\underbrace{H(\rho) := \int_{\mathbb{R}^k} \rho(x) \ln \rho(x) \mathrm{d}x}_{\textit{differential entropy}} \quad \text{and} \quad \underbrace{I(\rho) := \int_{\mathbb{R}^k} \rho(x) \|\nabla \ln \rho(x)\|^2 \mathrm{d}x}_{\textit{Fisher information}} \tag{1}$$

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Let c be a C^2 cost function.

Let $\mu_0, \mu_1 \in \mathcal{P}_{ac}(\mathbb{R}^d)$ be such that $H(\mu_i) < +\infty$.

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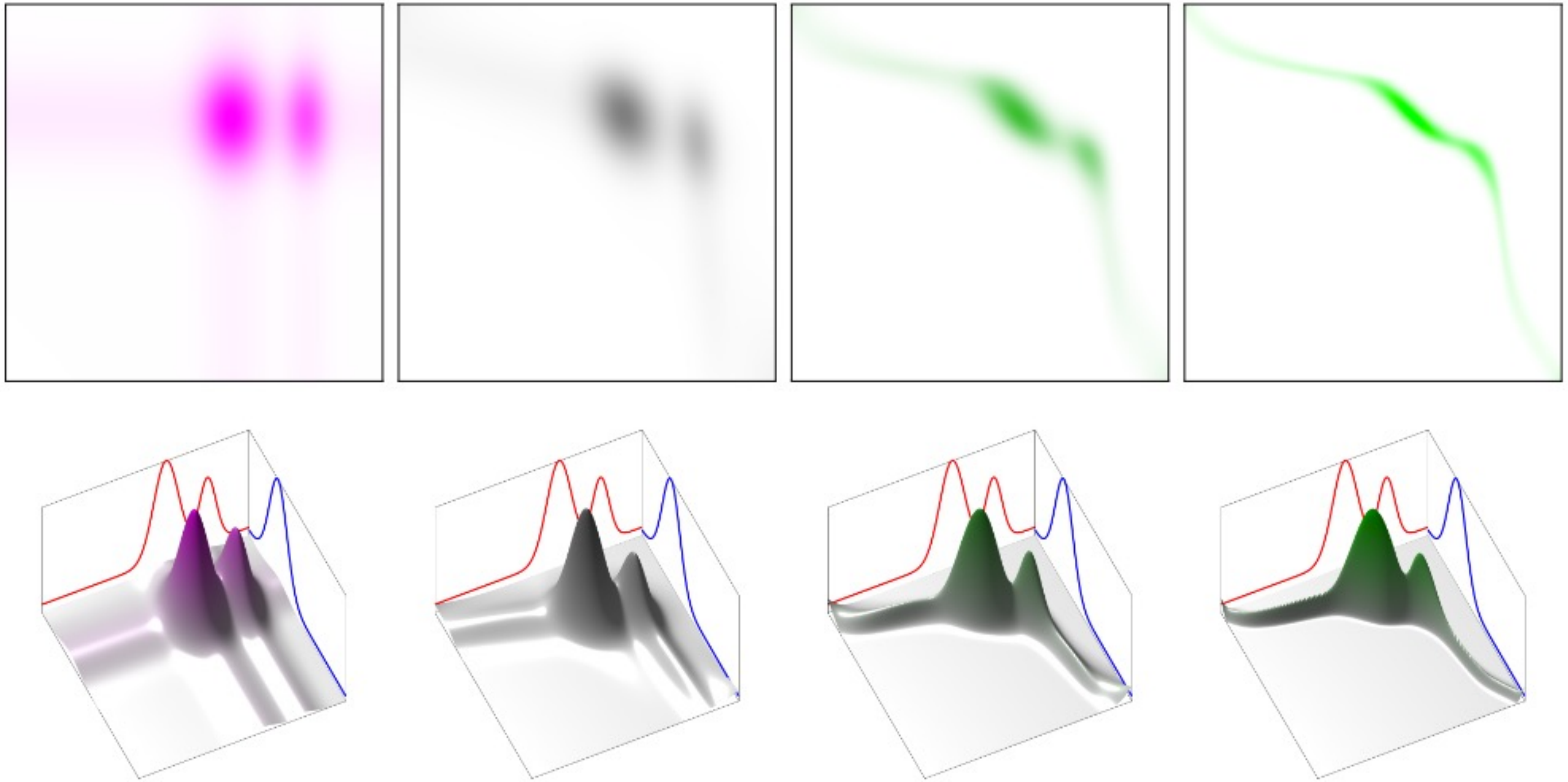
For $\varepsilon \geq 0$

$$OT_\varepsilon(\mu_0, \mu_1) := \inf_{\gamma \in \Pi(\mu_0, \mu_1)} \int c \mathrm{d}\gamma + \varepsilon H(\gamma) \tag{εEOT}$$



$\varepsilon = 10$ $\varepsilon = 1$ $\varepsilon = 10^{-1}$ $\varepsilon = 10^{-2}$

Peyré and Cuturi (2018)

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Question : What happens when $\varepsilon \to 0$?

Prior Works

Qualitative convergence results.

- Γ-convergence : [Mik04],[MT08],[Lé13],[CDPS15]

Quantitative convergence results.

- Discrete optimal transport : [CM94]
- Semi-discrete optimal transport : [ANWS21],[Del21]
- Finite Fisher information : [ADPZ11],[EMR15],[Con19]
- Finite entropy : [Pal19],[EN22],[CPT22]
- Multimarginal : [NP23]
- Sinkhorn divergence: [FSV+18, CRL+20]

Convergence of the value

Proposition [ADPZ11][EMR15]

Assume $c(x,y) = \frac{1}{2}\|x-y\|^2$, and that $Supp(\mu_i)$ are compact with $I(\mu_i) < +\infty$ then

$$OT_\varepsilon - OT_0 = -\frac{d}{2}\varepsilon \ln(2\pi\varepsilon) + \varepsilon\frac{H(\mu_0)+H(\mu_1)}{2} + o(\varepsilon) \qquad \text{(TE-OT}_\varepsilon\text{)}$$

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Proposition [EN22, CPT22]

Assume c is infinitesimally twisted and $Supp(\mu_i)$ compact then

$$\left(-\frac{d}{2}\varepsilon \ln(\varepsilon) + C'\varepsilon \leq\right) OT_\varepsilon - OT_0 \leq -\frac{d}{2}\varepsilon \ln(\varepsilon) + C\varepsilon \qquad (2)$$

Questions

Question 1 :

$$OT_\varepsilon - OT_0 = \underbrace{\int c \mathrm{d}\gamma_\varepsilon - \int c \mathrm{d}\gamma_0}_{suboptimality} + \varepsilon \underbrace{H(\gamma_\varepsilon)}_{entropy}$$

Can we disentangle ?

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Question 2 :

Is there a rate of convergence for $W_2(\gamma_\varepsilon, \gamma_0)$?

Fisher information and quadratic cost

Main result

Theorem [MS23]

Suppose that the cost is quadratic, that is $c(x, y) = \frac{1}{2}\|x - y\|^2$. Further assume that $I(\mu_i) < \infty$ and $Supp(\mu_i)$ compact. Then

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$$H(\gamma_\varepsilon) = -\frac{d}{2}\ln(2\pi\varepsilon) + H_m - \frac{d}{2} + o(1) \qquad (3)$$

where $H_m = \frac{H(\mu_0)+H(\mu_1)}{2}$.

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where $H_m = \frac{H(\mu_0)+H(\mu_1)}{2}$. Moreover

$$\int c d\gamma_\varepsilon - \int c d\gamma_0 = \frac{d}{2}\varepsilon + o(\varepsilon) \tag{4}$$

Sketch of proof

Recall $H_m = \frac{H(\mu_0)+H(\mu_1)}{2}$. The dynamic formulation [Lé13] is

$$OT_\varepsilon = \varepsilon H_m - \frac{d}{2}\varepsilon \ln(2\pi\varepsilon) + \min_{\substack{\partial\rho+\nabla\cdot(\rho v)=0\\ \rho_0=\mu_0,\rho_1=\mu_1}} \iint \frac{1}{2}|v_t|^2 \mathrm{d}\rho_t \mathrm{d}t + \frac{\varepsilon^2}{8}\int_0^1 I(\rho_t)\mathrm{d}t \qquad (\varepsilon\text{BB})$$

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$$\underbrace{\frac{1}{\varepsilon}\left(\iint \frac{1}{2}|v_t^\varepsilon|^2 \mathrm{d}\rho_t^\varepsilon \mathrm{d}t - OT_0\right)}_{\textit{suboptimality}} + \frac{\varepsilon}{8}\underbrace{\int_0^1 I(\rho_t^\varepsilon)\mathrm{d}t}_{\textit{regularity term}} = o(1) \qquad (6)$$

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Since both terms are positive they both tend to 0.

From dynamic to static and back

$$\underbrace{\int c\,\mathrm{d}\gamma_\varepsilon + \varepsilon H(\gamma_\varepsilon)}_{(a)\ static} = \underbrace{\varepsilon H_m - \frac{d}{2}\varepsilon \ln(2\pi\varepsilon)}_{(b)} + \underbrace{\iint \frac{1}{2}|v_t^\varepsilon|^2 \mathrm{d}\rho_t^\varepsilon \mathrm{d}t + \frac{\varepsilon^2}{8}\int_0^1 I(\rho_t^\varepsilon)\mathrm{d}t}_{(c)\ dynamic} \qquad (\varepsilon\mathrm{BB})$$

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Envelop theorem

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(a) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}(b) + \frac{\mathrm{d}}{\mathrm{d}\varepsilon}(c)$$

$$H(\gamma_\varepsilon) = H_m - \frac{d}{2}\ln(2\pi\varepsilon) - \frac{d}{2} + \frac{\varepsilon}{4}\int I(\rho_t^\varepsilon)\mathrm{d}t$$

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$$\begin{cases} \int c\mathrm{d}\gamma_\varepsilon - OT_0 = \iint \frac{1}{2}|v_t^\varepsilon|^2 \mathrm{d}\rho_t^\varepsilon \mathrm{d}t - OT_0 - \frac{\varepsilon^2}{8}\int I(\rho_t^\varepsilon)\mathrm{d}t + \frac{d}{2}\varepsilon \\ H(\gamma_\varepsilon) = \frac{\varepsilon}{4}\int_0^1 I(\rho_t^\varepsilon)\mathrm{d}t - \frac{d}{2}\ln(2\pi\varepsilon) + H_m - \frac{d}{2} \end{cases} \qquad (7)$$

Quadratic cost without Fisher information

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$$W_2(\gamma_\varepsilon, \gamma_0) \geq C\sqrt{\varepsilon}. \tag{9}$$

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Theorem [MS23]

Suppose that the cost is quadratic, that is $c(x,y) = \frac{1}{2}\|x-y\|^2$. Further assume that $\mu_i \in \mathcal{P}_{2+\delta,ac}$ for some $\delta > 0$ and that the Monge map ∇f is Lipschitz. Then

$$\int c \mathrm{d}\gamma_\varepsilon - \int c \mathrm{d}\gamma_0 = \Theta(\varepsilon), \quad H(\gamma_\varepsilon) = -\frac{d}{2}\ln(\varepsilon) + O(1) \tag{10}$$

$$C_1\sqrt{\varepsilon} \geq W_2(\gamma_\varepsilon, \gamma_0) \geq C_2\sqrt{\varepsilon} \tag{11}$$

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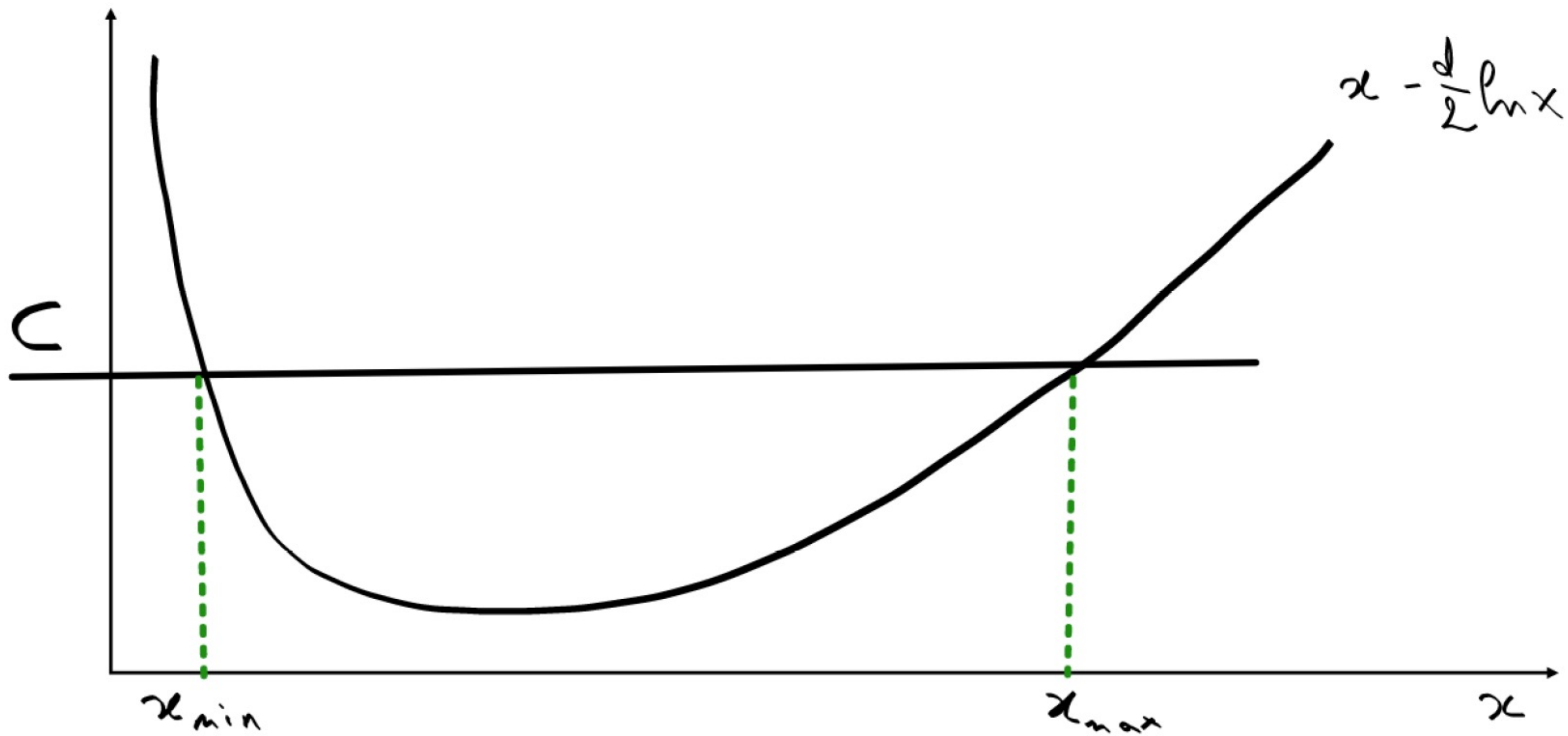
Combining both,

$$C \geq \frac{\int c\mathrm{d}\gamma_\varepsilon - \int c\mathrm{d}\gamma_0}{\varepsilon} - \frac{d}{2}\ln\left(\frac{\int c\mathrm{d}\gamma_\varepsilon - \int c\mathrm{d}\gamma_0}{\varepsilon}\right) \quad (15)$$



x - d/2 ln x
C
x_min
x_max
x

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the map $x \mapsto x - \frac{d}{2}\ln(x)$ is coercive, so

$$C_1\varepsilon \leq \int c d\gamma_\varepsilon - \int c d\gamma_0 \leq C_2\varepsilon$$

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We got an estimate of the type :

"If the suboptimality $\int c d\gamma - \int c d\gamma_0$ is small enough, then the entropy explodes"

$$H(\gamma) \gtrsim -\frac{d}{2} \ln\left(\int c d\gamma - \int c d\gamma_0\right)$$

Infinitesimally twisted costs and compact supports

Main result

Definition

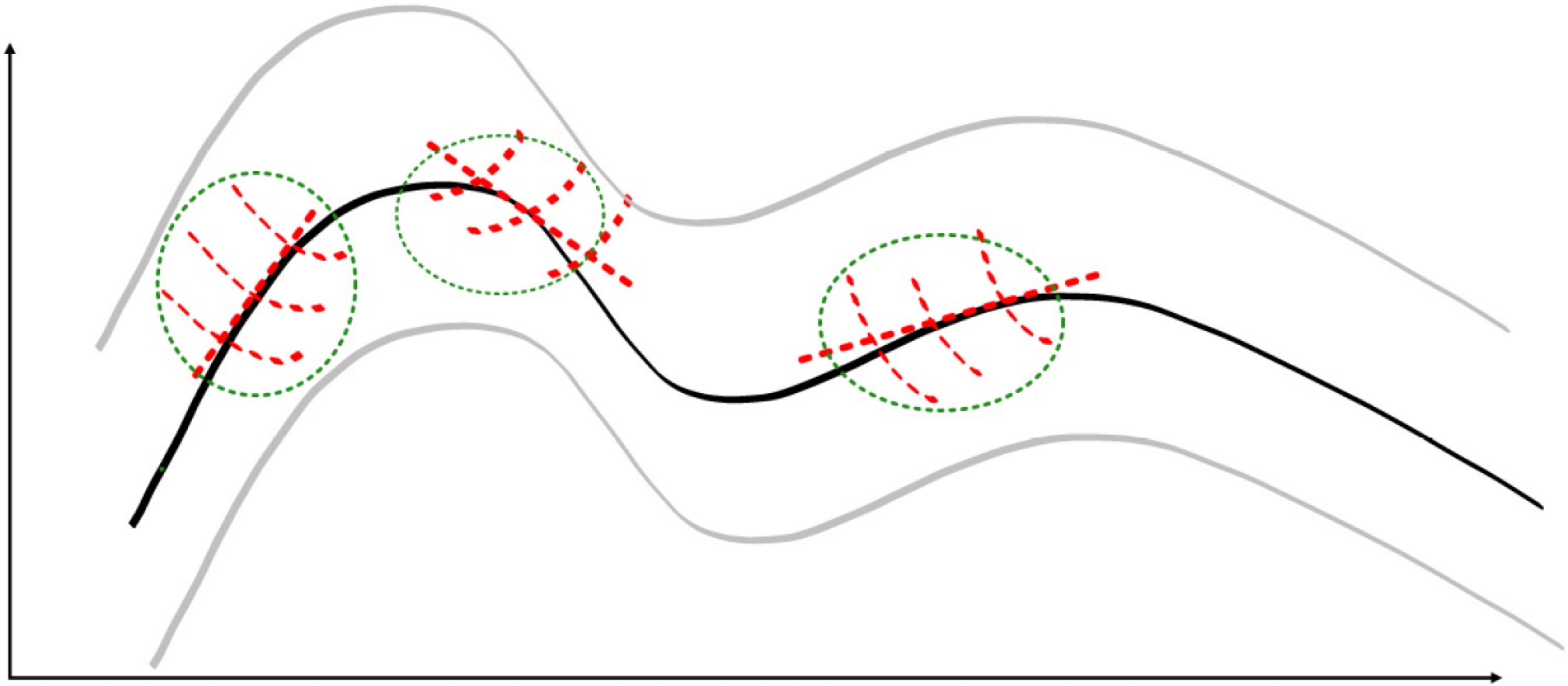
$c \in \mathcal{C}^2(\Omega^2)$ is said to be infinitesimally twisted if $\nabla^2_{xy} c(x,y) = (\partial^2_{x_i y_j} c(x,y))_{i,j} \in M_d(\mathbb{R})$ is invertible for every $(x,y) \in \Omega^2$.

Theorem

Suppose that the cost is $\mathcal{C}^2$ and infinitesimally twisted . Further assume that μ_i is compactly supported then

$$(c, \gamma_\varepsilon) = OT_0 + \Theta(\varepsilon), \quad H(\gamma_\varepsilon \mid \mathcal{H}^{2d}) = -\frac{d}{2}\ln(\varepsilon) + O(1), \quad \sqrt{\varepsilon} = O(W_2(\gamma_\varepsilon, \gamma_0)) \tag{16}$$

Note that here γ_0 is any optimal transport plan.



Thank you !

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