Derivation of a gyrokinetic model for electrons in the context of Hall-effect thrusters

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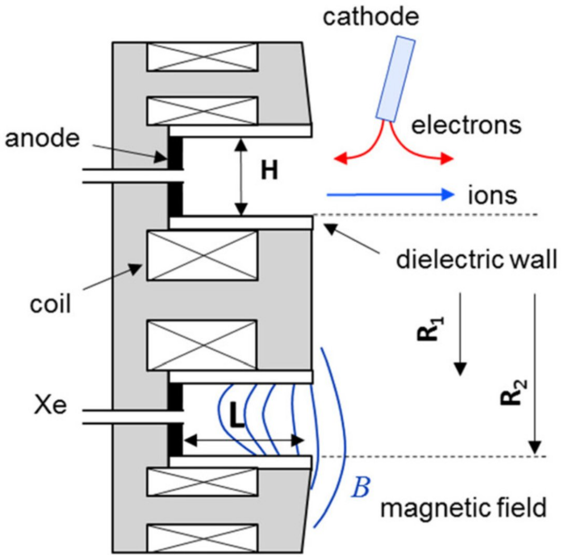
Projet CIEDS OPEN NUM DEF - AID

Outline

- Introduction
- Physical overview of the expected behaviour of electrons
- Kinetic modelling and scaling of the Boltzmann equation
- Analysis of the operator associated with the electro-magnetic force
- Derivation of the averaged kinetic equation

Context : Hall Thrusters



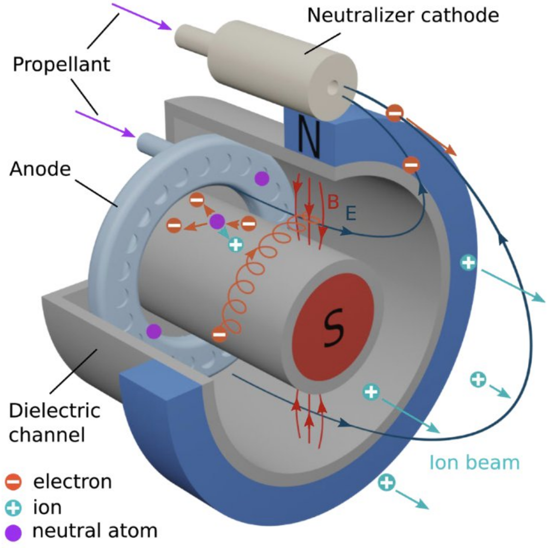


Figure 1 : Geometry of a Hall effect thruster, *J.P Boeuf 2017 (left)*, *Jakubczak et al 2021 (right)*

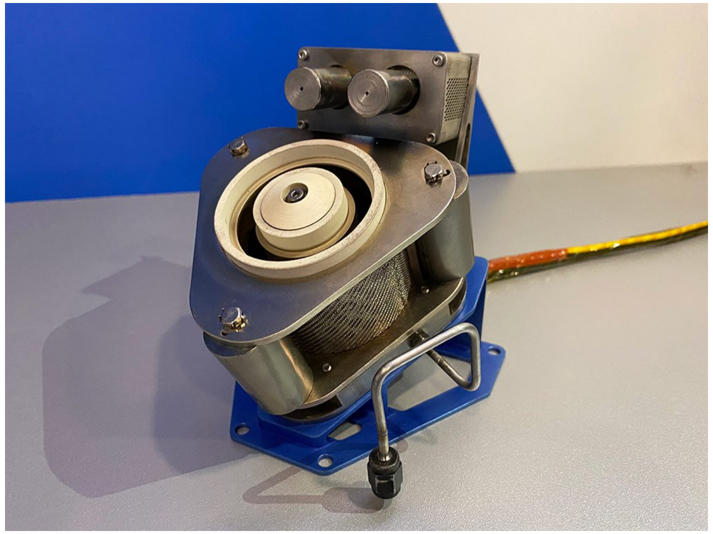


Figure 2 : General view of ST-40 thruster with two hollow cathodes

1) **Neutral gas injection**. e.g. Xenon
2) Cathode and anode: electron emission + Radial magnetic fields: generated by coils = **Trapped electrons**
3) **Gas ionization**: injected gas encounters electrons when it reaches the trapping zone
4) Potential difference: **accelerates ions after ionization**

Approach used in this presentation

- We focus on **electrons only**

- The fast gyration of the electrons leads us to adopt a **gyrokinetic approach :** we want to eliminate this time scale which induces **stiffness** in the equations.

Aim :
We want to average the rotation of the electrons around the magnetic line field ***to filter the rapid oscillations*** *of the system*

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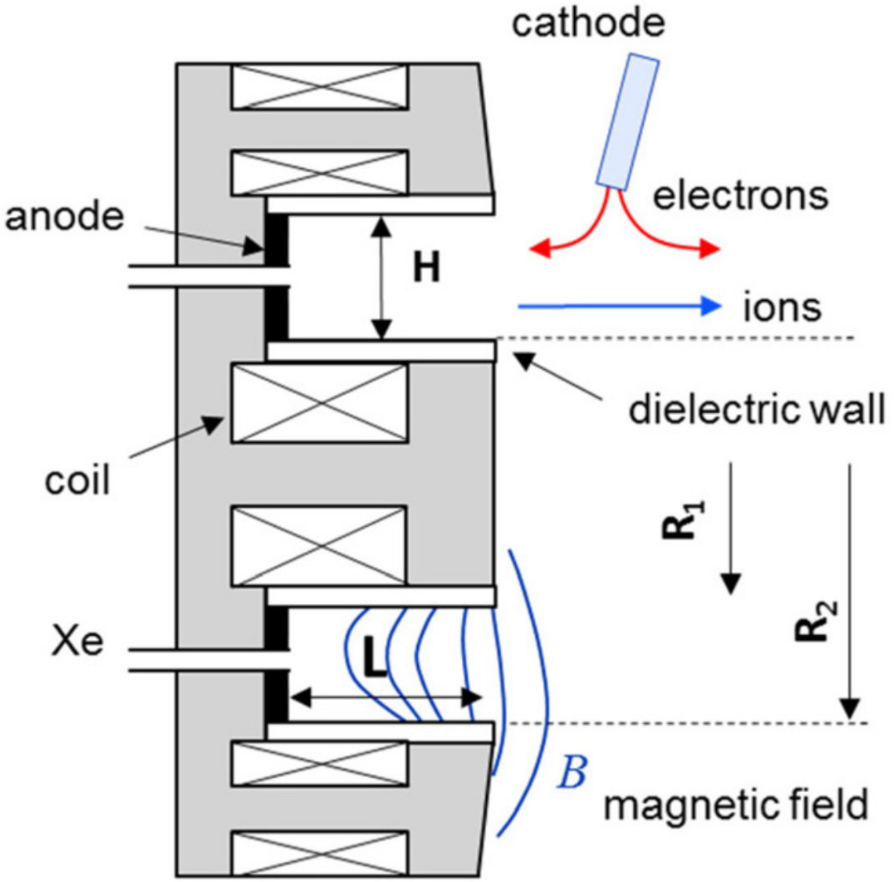
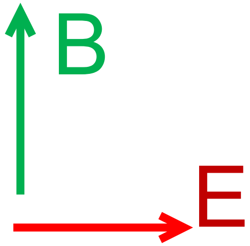
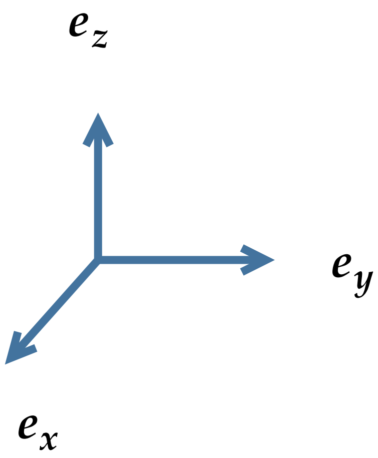
Geometry of the electromagnetic field







We consider the following magnetic field along e_z :

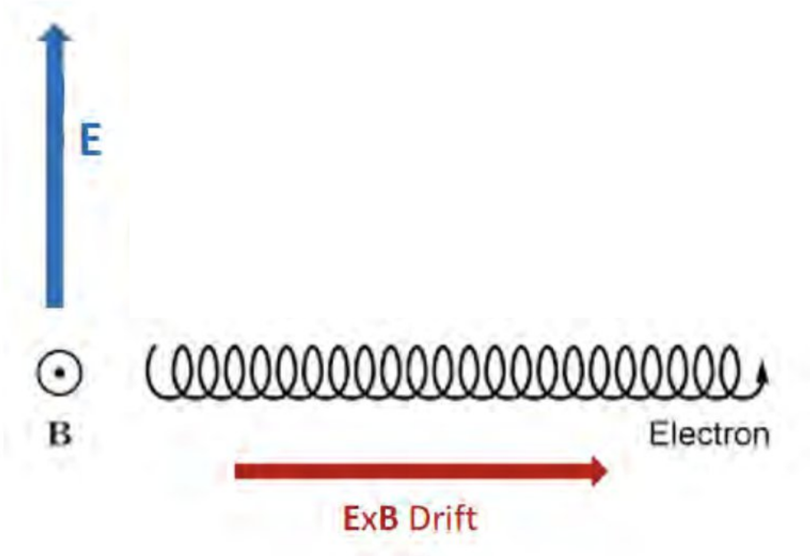
$$\boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ \overline{B} \end{pmatrix}$$

And an electric field that can be divided into a dominant field along e_y and a small perturbation :

$$\mathbf{E} = \overline{E} + \tilde{E}\,, \quad with \quad \overline{E} = \begin{pmatrix} 0 \\ \overline{E} \\ 0 \end{pmatrix} \; and \quad |\overline{E}| \gg |\tilde{E}|$$

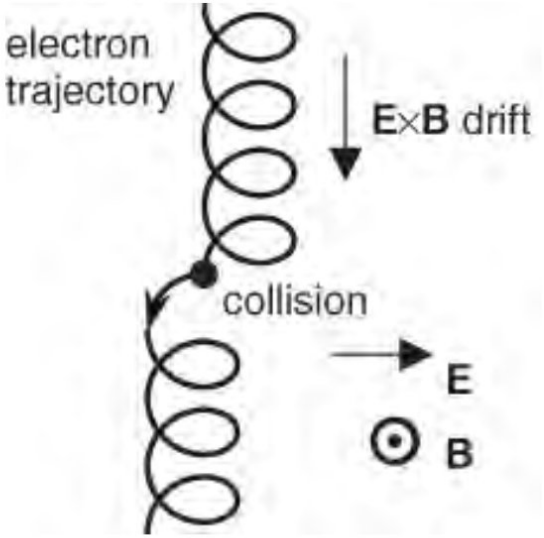
The fields do not depend on time

Physical intuiton of electrons trajectory



Trajectory of an electron immersed in electromagnetic field. We see the emergence of a drift in the ExB direction. We can show that it occurs at average speed :

$$v_D = \frac{|E|}{|B|} e_x$$



**taken from Phd Thesis of Sarah Sadouni*

In the presence of collisions (with other electrons or with the wall), the drift is disturbed. But we neglect the interactions with the walls and from litterature we expect to be in low collisional regime

<u>**Conclusion**</u> : The electrons should mainly drift in the ExB direction.

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Scaling of the equation

- Boltzmann equation for the electrons :

$$\partial_t f_e + \boldsymbol{v} \cdot \boldsymbol{\nabla}_x f_e - e[\boldsymbol{E} + \boldsymbol{v} \wedge \boldsymbol{B}] \cdot \boldsymbol{\nabla}_v f_e = Q(f_e, f_e)$$

- Mass ratio between electrons and heavy particles : $\epsilon = \sqrt{\frac{m_e}{m_n}} \ll 1$
- Inverse of the Hall parameter : $\beta \ll 1$ - because we are in a regime where the electrons are highly magnetised (trapped) -

with $\beta \ll \epsilon$

We then show that, in the regime considered, the Boltzmann equation for the electrons scales as follows:

$$\partial_t f_e + \frac{1}{\epsilon} \boldsymbol{v} \cdot \boldsymbol{\nabla}_x f_e - \frac{1}{\beta}[\bar{\boldsymbol{E}} + \boldsymbol{v} \wedge \boldsymbol{B}] \cdot \boldsymbol{\nabla}_v f_e - \tilde{\boldsymbol{E}} \cdot \boldsymbol{\nabla}_v f_e = Q(f_e, f_e)$$

electrons gyration >> transport >> others

Hilbert expansion in β

- Notation :

$$Tf_e = [\bar{\boldsymbol{E}} + \boldsymbol{v} \wedge \boldsymbol{B}] \cdot \boldsymbol{\nabla}_v f_e \qquad \text{and} \qquad Af_e = \boldsymbol{v} \cdot \boldsymbol{\nabla}_x f_e$$

- Ansatz :

$$f_e = \sum_i \beta^i f_e^{\,i}$$

We inject it in the Kinetic equation, and we get :

$$\begin{cases} Tf_e^{\,0} = 0 \\ \forall i \in \mathbb{N},\ Tf_e^{\,i+1} = \partial_t f_e^{\,i} + \frac{1}{\epsilon} Af_e^{\,i} - \tilde{\boldsymbol{E}} \cdot \boldsymbol{\nabla}_v f_e^{\,i} - Q(f_e^{\,i} f_e^{\,i}) \end{cases}$$

- This system shows the importance of the operator T in our problem : this motivates a precise study of its properties.

- In this presentation, we will focus on the main order of the expansion : $f_e^{\,0}$

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Characteristics lines of the operator T

Definition

Let $(x,v) \in \mathbb{R}^6$, let $s \in \mathbb{R}$, we define $(\mathbf{X}(s,\mathbf{x},\mathbf{v}), \mathbf{V}(s,x,v)) \in \mathbb{R}^6$ the characteristics lines of the operator T, with foot (x,v), as :

$$T f_e(t,\mathbf{X}(s,\mathbf{x},\mathbf{v}),\mathbf{V}(s,x,v)) = \frac{d}{ds} f_e(t,\mathbf{X}(s,\mathbf{x},\mathbf{v}),\mathbf{V}(s,x,v))$$

Property

The characteristics lines of the operator T are 2π-périodic and are given by :

$$\begin{cases} \mathbf{X}(s,\mathbf{x},\mathbf{v}) = x \\ \mathbf{V}(s,\mathbf{x},\mathbf{v}) = R(s)[v - v_D] + v_D \end{cases} \quad where \quad R(s) = \begin{pmatrix} cos(s) & -sin(s) & 0 \\ sin(s) & cos(s) & 0 \\ 0 & 0 & 1 \end{pmatrix} \; and \; v_D = \begin{pmatrix} \frac{\overline{E}}{\overline{B}} \\ 0 \\ 0 \end{pmatrix}$$

Note :

- The position characteristic is constant

- The velocity characteristic exhibits a composant v_D, which is the velocity that was assumed to drive the electrons, in the physical analysis of the problem

Average operator along the characteristic

Definition

We introduced $< . >$ the average operator along the characteristic curves of the operator T as :

$$< u > (t,\boldsymbol{x},\boldsymbol{v}) \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(\mathrm{t},\mathbf{X}(\mathrm{s},\mathbf{x},\mathbf{v}),\mathbf{V}(s,\boldsymbol{x},\boldsymbol{v}))ds$$

<u>Physical interpretation</u> : Guiding center approximation

$$< \mathbf{X}(\mathrm{s},\mathbf{x},\mathbf{v}) > = \mathbf{x} \text{ and } < \mathbf{V}(\mathrm{s},\mathbf{x},\mathbf{v}) > \;=\; \boldsymbol{v}_{\boldsymbol{D}}$$

The characteristics of the operator T are 2π-periodic and of mean velocity $\boldsymbol{v}_{\boldsymbol{D}}$, the drift velocity ExB

: we filter the oscillations of the system

Description of Ker T

Property

Let $u \in Ker(T)$, then u is **invariant** by the averaging operation ie :

$$< u > (t,x,v) = \frac{1}{2\pi}\int_0^{2\pi} u(t, \mathbf{X}(s,x,v), \mathbf{V}(s,x,v))\, ds = u(t,x,v)$$

• We now have a precise characterisation of the elements of the kernel of the operator T : these are the averaged functions.

• This is the « good » space to work with : the elements of this space are interpreted as the functions after averaging the fast gyration of the electrons.

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Order β^{-1}

First equation of the Hilbert expansion : $T f_e^{\,0} = 0$

Then we deduce that :

$$< f_e^{\,0} > (t,\boldsymbol{x},\boldsymbol{v}) = f_e^{\,0}(t,\boldsymbol{x},\boldsymbol{v})$$

• Actually we can show (by analysing the following orders) that in order to have uniqueness of the Hilbert expansion and of the solution, we can choose to impose that **the distributions of all the higher orders have zero mean**

• Therefore, by taking the average of the Hilbert expansion :

Property

At first order, the electron distribution function is the mean distribution obtained after filtering the oscillations along the characteristic lines of the operator T :

$$< f_e > = < \sum_i \beta^i f_e^{\,i} > = < f_e^{\,0} > = f_e^{\,0}$$

Order β^0

Second equation of the Hilbert expansion :

$$\partial_t {f_e}^0 + \frac{1}{\epsilon} A {f_e}^0 - T {f_e}^1 - \tilde{\boldsymbol{E}} \cdot \boldsymbol{\nabla}_v {f_e}^0 = Q\left({f_e}^0, {f_e}^0\right)$$

Goal : eliminate ${f_e}^1$ from the equation in order to obtain an equation that determine completely ${f_e}^0$

Idea : Introduce $\Pi_{\mathrm{Im(T)}^\perp}$ the projection operator on the orthogonal of Im(T) and apply it to the equation

$$\Pi_{\mathrm{Im(T)}^\perp}(\partial_t {f_e}^0) + \Pi_{\mathrm{Im(T)}^\perp}(\frac{1}{\epsilon} A {f_e}^0) - \underbrace{\cancel{\Pi_{\mathrm{Im(T)}^\perp}(T {f_e}^1)}}_{= 0 \text{ by definition}} - \Pi_{\mathrm{Im(T)}^\perp}(\tilde{\boldsymbol{E}} \cdot \boldsymbol{\nabla}_v {f_e}^0) = \Pi_{\mathrm{Im(T)}^\perp}(Q\left({f_e}^0, {f_e}^0\right))$$

Then we get an equation on ${f_e}^0$ only :

$$\boxed{\Pi_{\mathrm{Im(T)}^\perp}(\partial_t {f_e}^0) + \Pi_{\mathrm{Im(T)}^\perp}(\frac{1}{\epsilon} A {f_e}^0) - \Pi_{\mathrm{Im(T)}^\perp}(\tilde{\boldsymbol{E}} \cdot \boldsymbol{\nabla}_v {f_e}^0) = \Pi_{\mathrm{Im(T)}^\perp}(Q\left({f_e}^0, {f_e}^0\right))}$$

Order β^0

Problem : We need an expression of $\Pi_{\mathrm{Im(T)}^\perp}$ if we want to have a manageable equation on ${f_e}^0$

The last proposition is therefore an answer to this problem :

Property

Let $\Pi_{\mathrm{Im(T)}^\perp}$ the orthogonal projector on $\mathrm{Im(T)}^\perp$, we have the following equality :

$$\Pi_{\mathrm{Im(T)}^\perp} u \;=\; <u> \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(X(s,x,v),\, V(s,x,v))ds$$

Which then leads to the following equation :

$$<\partial_t {f_e}^0> \;+\; \frac{1}{\epsilon} <A {f_e}^0> \;-\; <\tilde{\boldsymbol{E}} \cdot \boldsymbol{\nabla}_v {f_e}^0> = \; <Q\left({f_e}^0, {f_e}^0\right)>$$

which can be developed by calculating

Order β^0

Theorem

The equation verified in the **weak sense** by ${f_e}^0$ is :

$$\partial_t {f_e}^0 + \frac{1}{\epsilon} div_x\left((\boldsymbol{v_D} + v_z \boldsymbol{e_z}) f_e^0 \right) + \frac{1}{\epsilon} div_v\left(\nabla_x \left(\frac{\boldsymbol{v_D}^2}{2} \right) f_e^0 \right) - \tilde{\boldsymbol{E}}_z \cdot \boldsymbol{\nabla}_v {f_e}^0 = < Q\left({f_e}^0, {f_e}^0 \right) >$$

with $\tilde{E}_z = \left(\tilde{E} \middle| e_z \right) e_z$ the part of the perturbation of the electric field which is collinear to the magnetic field, $\boldsymbol{v_D} = \begin{pmatrix} \frac{\overline{E}}{\overline{B}} \\ 0 \\ 0 \end{pmatrix}$

Interpretation :

$$\partial_t {f_e}^0 + \underbrace{\frac{1}{\epsilon} div_x\left((\boldsymbol{v_D} + v_z \boldsymbol{e_z}) f_e^0 \right)} + \underbrace{\frac{1}{\epsilon} div_v\left(\nabla_x \left(\frac{\boldsymbol{v_D}^2}{2} \right) f_e^0 \right)} - \underbrace{\tilde{\boldsymbol{E}}_z \cdot \boldsymbol{\nabla}_v {f_e}^0} = < Q\left({f_e}^0, {f_e}^0 \right) >$$

We have a drift-speed transort in the plane of the electron oscillations and a free transport in the orthogonal direction e_z

an acceleration imposed by the speed v_D which constraint the electrons

All the components of the electric field in the plane of the oscillations has vanished due to the filtration

Conclusion and future works

- Work achieved :

  - **a gyrokinetic model for modelling electrons** in the physical regime associated to Hall-effect thrusters : the **stiffness** associated with the part of the acceleration term **has been managed**.

- Work in progress :

  - Develop a **model of moments** for the electrons

  - Tackle **the stiffness on the space transport part** with for instance AP scheme (cf PHD thesis of Louis Reboul), Samurai library developped at CMAP etc