Asymptotic behavior of solutions of a nonlinear degenerate chemotaxis model

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Motivation

- asymptotic behaviour of solutions of the degenerate Keller-Segel system which is a nonlinear system modelling chemotaxis
- formation of pattern and rigorous mathematical description for the pattern dynamics of aggregating regions of biological individuals possessing the property of chemotaxis
- identify a destabilization mechanism that may lead to spatially non homogeneous solutions (Turing instability)

Our strategy is to consider any general perturbation of the solution nearby an homogenous steady state, we prove that its nonlinear evolution is dominated by the corresponding linear dynamics along the finite number of fastest growing modes.

Outline

1. Pattern Formation
   - Turing Principle
   - The model of Keller-Segel
   - The instability Criterion
   - Linear Instability
   - Nonlinear Instability
2. Finite Volume Method
   - FV Discretization
3. Numerical results
   - The Heterogeneous State
   - The Numerical Test: Test 2

Patterns are everywhere

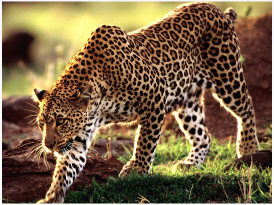
Why do leopards have spots and tigers have stripes?

Patterns are the solutions of a reaction-diffusion system which are stable in time and stationary inhomogeneous in space.

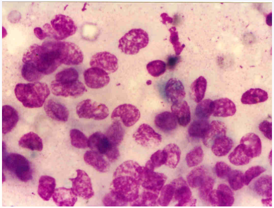
Pattern formation: homogeneous steady states lose stability to stable inhomogeneous solutions

Animal skin: stripes

Sand dunes

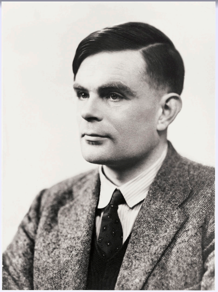


Animal skin: spots



Tumor cells

Patterns = Alan Turing (1912–1954)



Alan Turing

Models for two chemical species, U (activator) and V (inhibitor)

Reaction-Diffusion System

$$\begin{cases} \partial_t U = \Delta U + \gamma f(U,V) & \text{in } Q_T, \\ \partial_t V = \underbrace{d\Delta V}_{\text{diffusion}} + \gamma \underbrace{g(U,V)}_{\text{reaction}} & \text{in } Q_T. \end{cases}$$

γ : is proportional to the area (scale parameter)
d : diffusion coefficient

- A kinetic system of chemicals, stable, in the absence of diffusion (ODE), becomes unstable in the presence of diffusion (PDE).
- Stabilizing reaction kinetics + diffusion (stabilizing) $\longrightarrow$ instability!
- Diffusion driven pattern formation (nowadays: Turing patterns).
- Counter intuitive: Diffusion was/is thought of having stabilizing effect.

A. Turing, The chemical basis of morphogenesis, *Biological Sciences* 237 37-72, 1952.

The model of Keller-Segel

A typical model describing chemotaxis is the Keller-Segel model : It is the most popular model to the chemical control of the cell movement.

E.F. Keller and L.A. Segel. The Keller-Segel model of chemotaxis (1970).

• Evolution of cell density (u) :

$$\partial_t u - \overbrace{div(a(u)\nabla u)}^{\text{diffusion term}} + \overbrace{div(\chi(u)\nabla v)}^{\text{chemotaxis term}} = 0$$

• Evolution of the concentration of the chemoattractant (v):

$$\partial_t v - d\Delta v = \underbrace{\alpha u - \beta v}_{\text{production and death}}$$

- $a(u)$: the diffusivity of the cells (mobility)
- $\chi(u)$: the chemotactic sensitivity of cells towards the chemoattractant.
- the rate given by α modeling the production of chemoattractant to attract the other cells.

The model of Keller-Segel

- Evolution of cell density (u) :

$$\partial_t u - \overbrace{div(a(u)\nabla u)}^{\text{diffusion term}} + \overbrace{div(\chi(u)\nabla v)}^{\text{chemotaxis term}} = 0$$

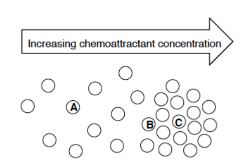
- Evolution of the concentration of the chemoattractant (v):

$$\partial_t v - d\Delta v = \underbrace{\alpha u - \beta v}_{\text{production and death}}$$

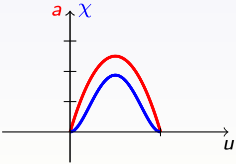
Volume-Filling Effect : The particles are assumed of having a finite volume and the cells cannot move into regions that are already filled by other cells.

Degenerate diffusion and sensitivity

$$a(u) = a_0 u(1-u), \quad \chi(u) = \chi_0 u^2 (1-u)^2.$$



Volume-Filling Effect



Homogeneous steady state: Keller-Segel

$$\begin{cases} \partial_t U - div(a(U)\nabla U) + div(\chi(U)\nabla V) = 0 \\ \partial_t V - d\Delta V = \alpha U - \beta V \end{cases} \tag{1}$$

Consider a uniform constant solution (when the diffusion terms are neglected) that forms a homogeneous steady state verifying

$$\alpha \overline{U} = \beta \overline{V}, \quad 0 < \overline{U} < 1$$

Our target is to study the nonlinear evolution of a perturbation around the homogeneous steady state

$$u(\mathbf{x}, t) = U(\mathbf{x}, t) - \overline{U}, \qquad v(\mathbf{x}, t) = V(\mathbf{x}, t) - \overline{V}$$

The nonlinear evolution $(u(\mathbf{x}, t), v(\mathbf{x}, t))$ satisfies the equivalent system

$$\begin{cases} \partial_t u - \nabla \cdot \left(a(u + \overline{U})\nabla u \right) + \nabla \cdot \left(\chi(u + \overline{U})\nabla v \right) = 0 \\ \partial_t v - d\,\Delta v = \alpha u - \beta v \end{cases} \tag{2}$$

Matrix From

The system can be written in a matrix form

$$\partial_t W = \underbrace{\left(\overline{D}\Delta W + \mathcal{A}W\right)}_{\text{Linear operator}} + \underbrace{\left(\nabla\cdot(D\nabla W) - \overline{D}\Delta W - \mathcal{A}W + \mathcal{F}\right)}_{\text{Nonlinear operator}} = \mathcal{L}(W) + \mathcal{N}(W)$$

where,

$$W(\mathbf{x},t) = \begin{pmatrix} u(\mathbf{x},t) \\ v(\mathbf{x},t) \end{pmatrix}, \quad \overline{D} = \begin{pmatrix} a(\overline{U}) & -\chi(\overline{U}) \\ 0 & d \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & 0 \\ \alpha & -\beta \end{pmatrix}.$$

$$D = \begin{pmatrix} a(u+\overline{U}) & -\chi(u+\overline{U}) \\ 0 & d \end{pmatrix}, \quad \mathcal{F} = \begin{pmatrix} 0 \\ \alpha u - \beta v \end{pmatrix}.$$

The corresponding **linearized** Keller-Segel system $W_L = (u_L, v_L)$ takes the form:

$$\begin{cases} \partial_t u_L = a\left(\overline{U}\right)\Delta u_L - \chi\left(\overline{U}\right)\Delta v_L, \\ \partial_t v_L = d\Delta v_L + \alpha u_L - \beta v_L. \end{cases} \tag{3}$$

Linear Instability Criterion : $Re(\lambda_q) > 0$ for some $q \neq 0$

Let $\Omega = \mathbb{T}^n = \prod_{i=1}^{n}]0, \pi[$ be a n-dimensional box for $n = 1, 2$ or 3. The corresponding **linearized** Keller-Segel system then takes the form:

$$\begin{cases} \partial_t u_L = a\left(\overline{U}\right) \Delta u_L - \chi\left(\overline{U}\right) \Delta v_L, \\ \partial_t v_L = d \Delta v_L + \alpha u_L - \beta v_L. \end{cases} \tag{4}$$

Let $q = (q_1, ..., q_n) \in \mathbb{N}^n$ and let

$$e_q(\mathbf{x}) = \Pi_{i=1}^{n} \cos(q_i x_i).$$

Then, $\{e_q\}_{q \in \mathbb{N}^n}$ forms a Hilbert basis of the space of functions in $L^2(\Omega)$.
We look for a normal mode to the linear Keller-Segel system (4) of the following form:

$$W_L(\mathbf{x}, t) = [u_L(\mathbf{x}, t), v_L(\mathbf{x}, t)] = \exp(\lambda_q t)\, e_q(\mathbf{x})\, \vec{r}_q, \text{ where } \vec{r}_q \text{ is an eigenvector.} \tag{5}$$

Plugging equation (5) into system (4) yields

$$\lambda_q \vec{r}_q = \begin{bmatrix} -a\left(\overline{U}\right) \|q\|^2 & \chi\left(\overline{U}\right) \|q\|^2 \\ \alpha & -d\, \|q\|^2 - \beta \end{bmatrix} \vec{r}_q := \mathcal{B}\, \vec{r}_q, \text{ where } \|q\|^2 = \sum_{i=1}^{n} q_i^2.$$

A nontrivial normal mode can be obtained by setting

$$det \begin{bmatrix} \lambda_q + a\left(\overline{U}\right) \|q\|^2 & -\chi\left(\overline{U}\right) \|q\|^2 \\ -\alpha & \lambda_q + d\, \|q\|^2 + \beta \end{bmatrix} = 0.$$

Linear Instability Criterion : $Re(\lambda_q) > 0$ for some $q \neq 0$

The corresponding dispersion relation associated to the linearised system is:

$$P(\lambda_q) = \lambda_q^2 + \underbrace{\left\{\|q\|^2 \left(d + a\left(\overline{U}\right)\right) + \beta\right\}}_{>0} \lambda_q + \underbrace{\|q\|^2 \left\{a\left(\overline{U}\right)\left(d\,\|q\|^2 + \beta\right) - \chi\left(\overline{U}\right)\alpha\right\}}_{:=h\left(\|q\|^2\right)} = 0. \quad (4)$$

The discriminant is positive, then we deduce a linear instability by requiring:

Linear Instability condition

$$h\left(\|q\|^2\right) = det\left(\mathcal{B}\right) = \|q\|^2 \left\{a\left(\overline{U}\right)\left(d\,\|q\|^2 + \beta\right) - \chi\left(\overline{U}\right)\alpha\right\} < 0. \quad (5)$$

Therefore, we can denote two distinct real roots for all q by

$$\lambda_q^{\pm} = -\frac{1}{2}\left(\|q\|^2\left(d + a\left(\overline{U}\right)\right) + \beta\right) \pm \frac{1}{2}\sqrt{\tilde{D}}.$$

We have $\lambda_q^- < 0$. And, one has $a\left(\overline{U}\right)\left(d\,\|q\|^2 + \beta\right) - \chi\left(\overline{U}\right)\alpha > 0$, for q large. Hence, there are only finite numbers of q such that $h\left(\|q\|^2\right) < 0$ and $\lambda_q^+ > 0$.
Define the largest eigenvalue $\lambda_{\max} > 0$ and

$$Q_{max} = \{q \in \mathbb{N}^n \text{ such that } \lambda_q^+ = \lambda_{\max}\}.$$

Linear Instability Criterion : $Re(\lambda_q) > 0$ for some $q \neq 0$

The corresponding linearly independent eigenvectors are given by

$$r_q^{\pm} = \begin{pmatrix} (\lambda_q^{\pm} + d\,\|q\|^2 + \beta)/\alpha \\ 1 \end{pmatrix}. \tag{4}$$

Transition From Linear to Nonlinear Instability I

Theorem (Weakly asymptotic behavior: Degenerate case)

Assume the instability criterion and given an initial perturbation

$$W(\mathbf{x},0) = \sum_{q\in\mathbb{N}^n} \left\{ w_q^- r_q^- + w_q^+ r_q^+ \right\} e_q(\mathbf{x}) \in L^2(\Omega),$$

Then, there exists three positive constants C, ν, and K_1 such that

$$\left\| e^{-\lambda_{max} t} W(\mathbf{x},t) - \sum_{q\in Q_{\max}} w_q^+ r_q^+ e_q(\mathbf{x}) \right\|_\star \leq C \|W_0\|_{L^2(\Omega)} \left\{ e^{-\nu t} + e^{K_1 t} \right\}, \text{ for all } t > 0$$

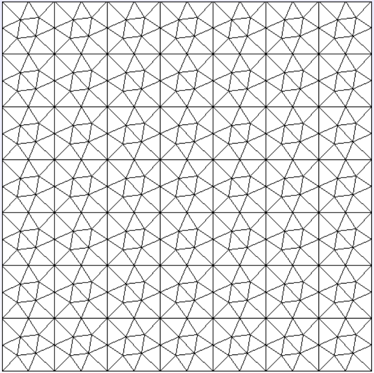
with $Q_{max} = \{q \in \mathbb{N}^n;\ \lambda_q^+ = \lambda_{max}\}$, $\nu = \min_{q\notin Q_{max}} |\lambda_{max} - \lambda_q| > 0$ is the gap between λ_{max} and the rest and $\|w\|_\star = \|(-\Delta)^{-\frac{1}{2}} u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}$,

Idea of proofs.

- The solution of the chemoattractant concentration is regular.
- Consider $(-\Delta)^{-1} u$ and v as test functions to compare the nonlinear system and the linearised one.
- Compare the nonlinear system to the linear fastest growing modes.

Chamoun, M. Ibrahim, M. Saad, R. Talhouk, Asymptotic behavior of solutions of a nonlinear degenerate chemotaxis model. Discrete and Continuous Dynamical Systems, 2020.

The Mesh



Portion of the triangular mesh for the space domain $\Omega = (0,1) \times (0,1)$ with 14336 acute angle triangles.

Fixed step time $\Delta t = 0.01$.

The mesh satisfy the orthogonality condition, and $\Delta x \approx 0.01$.

Data and Bifurcation: Test 1

To determine explicitly the critical value, we fix all parameters of system except the chemotactic sensitivity.

$$\begin{cases} \partial_t U - \nabla \cdot (a(U)\nabla U) + \nabla \cdot (\chi(U)\nabla v) = 0 \\ \partial_t V - d_V \Delta V = \alpha U - \beta V \end{cases}$$

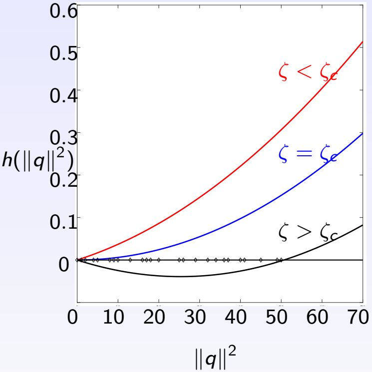
Steady State: $\alpha\overline{U} = \beta\overline{V}$.
Data: $\alpha = 5$, $\beta = 11$, $a(U) = d_U (U(1-U))^2$, $\chi(U) = \zeta \underbrace{U(1-U)}_{:=\varphi(U)}$, and $d_U = 0.001$, $d_V = 0.01$.
Linear instability:

$$h\left(\|q\|^2\right) = \|q\|^2 \left\{ a\left(\overline{U}\right) d \|q\|^2 + a\left(\overline{U}\right)\beta - \zeta\varphi\left(\overline{U}\right)\alpha \right\} < 0. \tag{5}$$

the critical chemosensitivity ζ_c is given by $\zeta_c = \frac{a(\overline{U})\beta}{\varphi(\overline{U})\alpha}$.

For $\overline{U} = 0.5$, we evaluate the critical chemosensitivity and we get $\zeta_c = 10^{-4}$.

Data and Bifurcation: Test 1



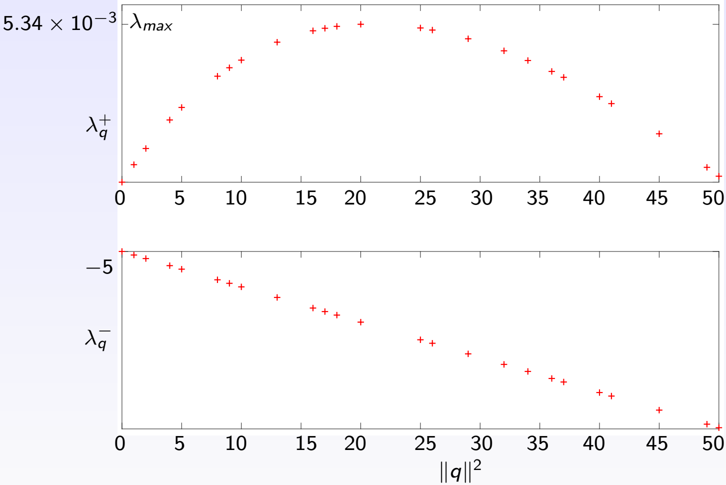
There is a critical value ζ_c such that

- **No pattern formation if ζ is below this critical value ζ_c**
- **Pattern formation can be expected if ζ is somewhere else above the critical value**
- When the chemosensitivity strength ζ increases beyond the critical value ζ_c, a **finite range of unstable wave numbers** $\|q\|^2$ exist.

Range of unstable wave numbers:

$$0 < \|q\|^2 < \underbrace{-\frac{a\left(\overline{U}\right)\beta - \chi\left(\overline{U}\right)\alpha}{a\left(\overline{U}\right)d}}_{:=\|q^\star\|^2}.$$

Positive and Negative Eigenvalues: $\zeta = 10^{-3} > \zeta_c$.



Distribution of positive eigenvalues λ_q^+ and negative eigenvalues λ_q^- with respect to the range of unstable wave numbers $\|q\|^2$. We have $\|q^\star\|^2 = 50, \quad \|q\|^2_{\max} = 20$

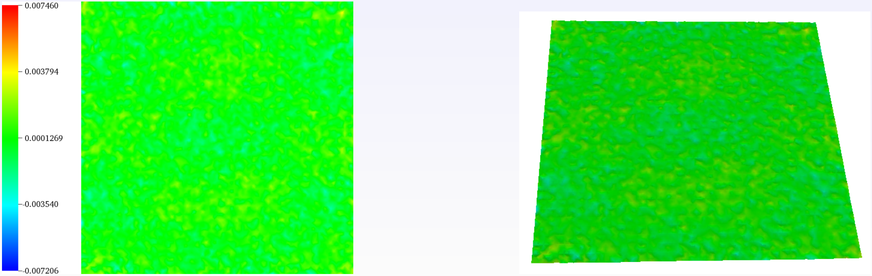
The set of maximal wave numbers : $Q_{max} = \{(2,4),(4,2)\}$.
The number of unstable wave number is 42.

Initial Condition for the Cell Density

Consider a perturbation with an order of magnitude equal to 8×10^{-2} around the steady state $\overline{U}$ given by:

$$W_0(\mathbf{x}) = \sum_{q \in \mathbb{N}^n ; h(\|q\|^2) \leq 0} \delta_q \left\{ w_q^- r_q^- + w_q^+ r_q^+ \right\} e_q(\mathbf{x}) \in L^2(\Omega), \quad \delta_q \text{ a randum number}$$

$U(0, x) = \overline{U} + W_0$

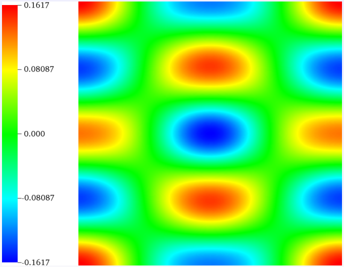


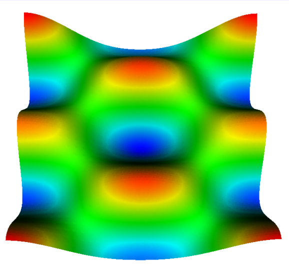
Initial perturbation around $\overline{U}$

Heterogeneous State:=Pattern

We want to show that the behavior of the nonlinear evolution is similar to a heterogeneous stationary solution given by

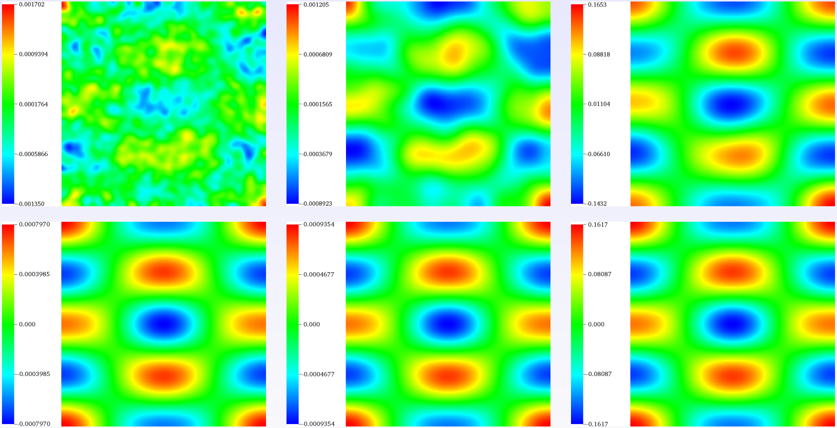
$$e^{\lambda_{max} t} \sum_{q \in Q_{\max}} w_q^+ r_q^+ e_q(\mathbf{x})$$





This solution can be computed based on the bifurcation analysis without any simulation of the nonlinear Keller-Segel system.

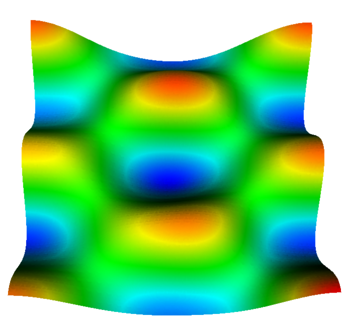
Spatial Evolution of $u(\mathbf{x}, t)$ vs Heterogeneous State : Test 1

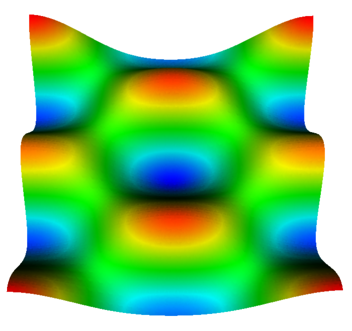


First row from left to right: Nonlinear evolution of the function $u(\mathbf{x}, t)$ at $t = 2.5$, $t = 325$, and $t = 997.5$.

Second row from left to right: Evolution of the heterogeneous state at the same moments as for the evolution of $u(\mathbf{x}, t)$.

Spatial Evolution of $u(\mathbf{x},t)$ vs Heterogeneous State: Test 1

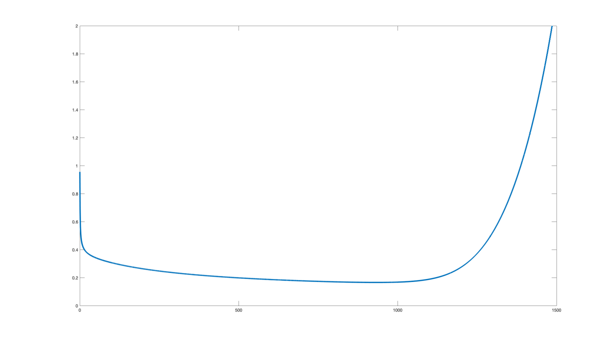




Similarities of patterns between the nonlinear evolution $u(\mathbf{x},t)$ (to the left) and the heterogeneous state (to the right). The L^2 error norm is 0.02 at time T=1000.

Time Evolution of $u(\mathbf{x}, t)$ vs Heterogeneous State: Test 1

$$E(t) = \frac{\left\| W(\mathbf{x}, t) - e^{\lambda_{max} t} \sum_{q \in Q_{\max}} w_q^+ r_q^+ e_q(\mathbf{x}) \right\|_{L^2(\Omega)}}{\| W(\mathbf{x}, t) \|_{L^2(\Omega)}}$$



The two solutions are very close for a long time but when the nonlinear solution reaches its final state by forming pattern, the expected solution with the exponential continues to grow. For that reason, there is a critical time for which the theorem is valid depending on initial conditions.
The relative error increases after $T^{\delta} \approx 1000$ since the nonlinear solution is steady stable and $e^{\lambda_{max} t} \to \infty$.

Data and Bifurcation: Test 2

To determine explicitly the critical value, we fix all parameters of system except the death rate.

$$\begin{cases} \partial_t U - \nabla \cdot (a(U)\nabla U) + \nabla \cdot (\chi(U)\nabla v) = 0 \\ \partial_t V - d_V \, \Delta V = \alpha U - \beta V \end{cases}$$

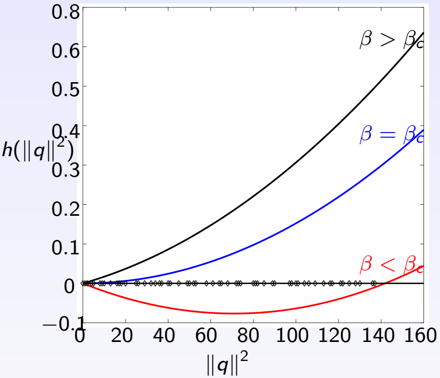
Steady State: $\alpha\overline{U} = \beta\overline{V}$.
Data:
In this numerical test, we fix $\alpha = 2$, $a(U) = d_U (U(1-U))^3$, $d_U = 0.01$, $\chi(U) = \zeta (U(1-U))^3$, $\zeta = 0.001$ and $d_V = 0.01$.
Linear instability:

$$h\left(\|q\|^2\right) = \|q\|^2 \left\{ a\left(\overline{U}\right) d \, \|q\|^2 + a\left(\overline{U}\right)\beta - \zeta\varphi\left(\overline{U}\right)\alpha \right\} < 0. \tag{5}$$

the critical death β_c is given by $\beta_c = \frac{\zeta\varphi(\overline{U})\alpha}{a(\overline{U})}$.

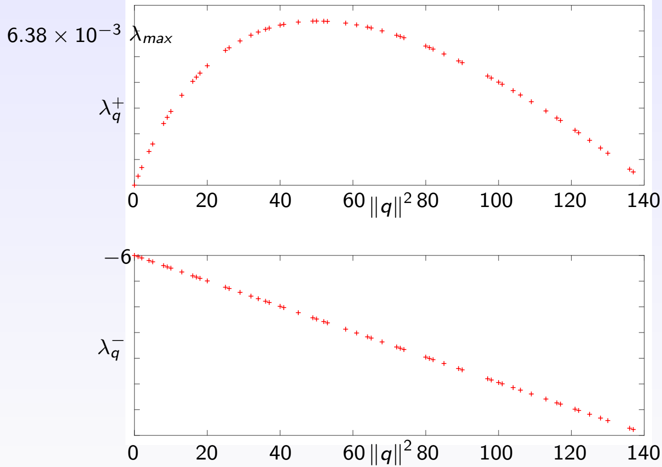
For $\overline{U} = 0.5$, we evaluate the critical chemosensitivity and we get $\beta_c = 20$.

Data and Bifurcation: Test 2



- No pattern formation if *beta* is above this critical value β_c
- Pattern formation can be expected if β is somewhere else below the critical value
- When the death rate β decreases beyond the critical value β_c, a finite range of unstable wave numbers $\|q\|^2$ exist.

Positive and Negative Eigenvalues: $\beta = 6 > \beta_c$



To the top: Distribution of positive eigenvalues λ_q^+ with respect to the range of unstable wave numbers $\|q\|^2$. *To the bottom:* Distribution of negative eigenvalues λ_q^- with respect to the range of unstable wave numbers $\|q\|^2$.

$\|q^\star\|^2 = 141$, $\|q_{max}\|^2 = 50$ The set of wave numbers max : $Q_{max} = \{(1,7),(5,5),(7,1)\}$.

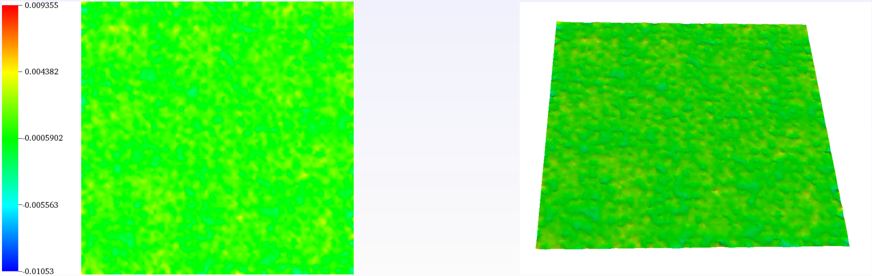
The number of unstable wave number is 58.

Initial Condition for the Cell Density: Test 2

Consider a perturbation with an order of magnitude equal to 10^{-2} around the steady state $\overline{U}$ given by:

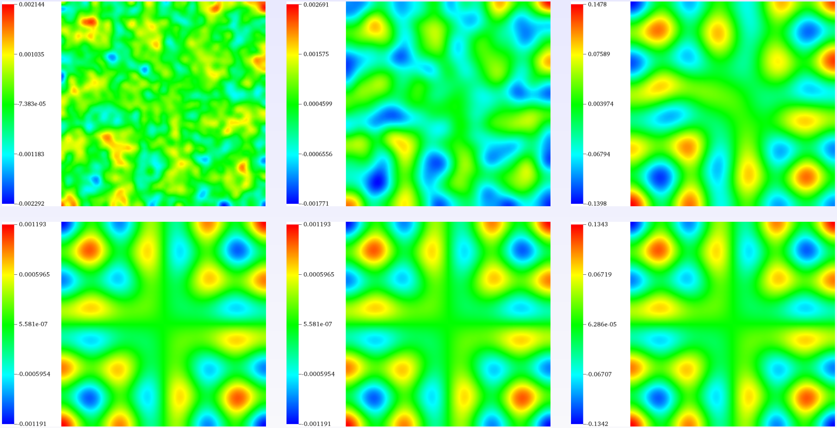
$$W_0(\mathbf{x}) = \sum_{q \in \mathbb{N}^n ; h(\|q\|^2) \leq 0} \delta_q \left\{ w_q^- r_q^- + w_q^+ r_q^+ \right\} e_q(\mathbf{x}) \in L^2(\Omega), \quad \delta_q \text{ a randum number}$$

$U(0, x) = \overline{U} + W_0$



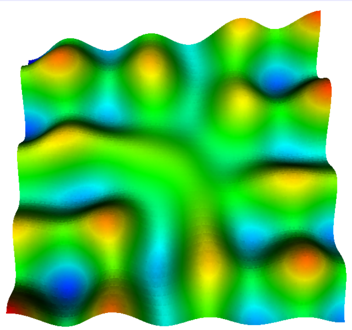
Initial condition of the function $u(\mathbf{x}, t)$ with a small perturbation around zero. 2D view of the function $u(\mathbf{x}, t)$ *(to the left) and a 3D view of its magnitude (to the right).*

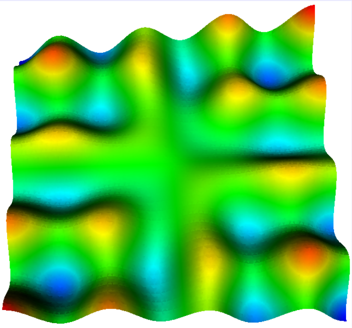
Spatial Evolution of $u(\mathbf{x}, t)$ vs Heterogeneous State: Test 2



First row from left to right. Nonlinear evolution of the function $u(\mathbf{x}, t)$ at $t = 10$, $t = 70$, and $t = 750$. *Second row from left to right.* Evolution of the heterogeneous stationary solutions at the same moments as for $u(\mathbf{x}, t)$.

Spatial Evolution of $u(\mathbf{x}, t)$ vs Heterogeneous State: Test 2





Similarities of patterns between the nonlinear evolution $u((\mathbf{x}, t)$ (to the left) and the heterogeneous state (to the right).

Conclusion

- Initial condition is prepared to be a perturbation around the steady state
- Study of linearised system to be able to the determine bifurcation parameters
- A lot of parameters to be determined
- The nonlinear evolution is dominated by the corresponding linear dynamics along the finite number of fastest growing modes