Time-Convolutionless Master Equation Applied to Adiabatic Elimination

CANUM *Ile-de-Ré*

Angela Riva

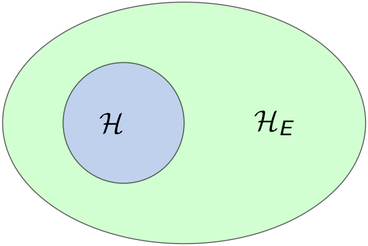
Quantic team,
ENS-PSL, CNRS, Inria,
Mines Paris, Université PSL

Masaaki Tokieda

Department of Chemistry,
Graduate School of Science,
Kyoto University

May 30, 2024

Model reduction for open quantum systems

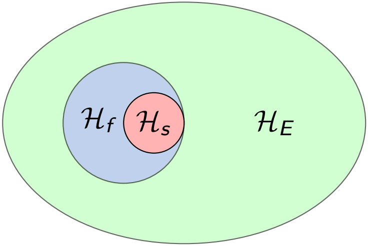


$$\mathcal{H}_{\text{TOT}} = \mathcal{H} \otimes \mathcal{H}_E$$

By tracing out the environment, an **effective model** of the open quantum system dynamics

$$\frac{d}{dt}\rho = \mathcal{L}\rho$$

Model reduction for open quantum systems



$$\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_f$$

Similarly, we can obtain a further **reduced model** $\vec{x}$ of the dynamics for a subsystem

$$\frac{d}{dt}\rho = \mathcal{L}\rho \quad \rightarrow \quad \frac{d}{dt}\vec{x} = \mathcal{F}\vec{x}$$

A reduced model of an OQS is useful

- To simulate the **effective dynamics** of a subsystem.

  Computing the full Lindblad time evolution is hard:

  if $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \implies \dim \mathcal{H} = \dim \mathcal{H}_1 \cdot \dim \mathcal{H}_2 \cdot ...$

A reduced model of an OQS is useful

- To simulate the **effective dynamics** of a subsystem.

  Computing the full Lindblad time evolution is hard:

  if $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \implies \dim \mathcal{H} = \dim \mathcal{H}_1 \cdot \dim \mathcal{H}_2 \cdot \ldots$

- To **engineer** a desired coupling to **dissipative reservoir**.

  Bosonic codes (**cat qubits**, GKPs...) rely on the autonomous stabilization of the qubit encoding subspace.

Encoding a cat-qubit in a harmonic oscillator

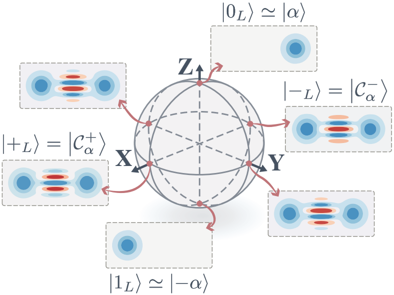


Fig. from: R. Gautier, A. Sarlette, M. Mirrahimi, PRX Quantum 2022

$|\pm\alpha\rangle$ coherent states localised on opposite sides of the phase space

$$|\mathcal{C}_\alpha^\pm\rangle = (|+\alpha\rangle \pm |-\alpha\rangle)/\mathcal{N}_\pm$$

Local noise cannot bring one logical state to the other.

Cochrane et al., PRA 1999

Engineering the cat-qubit confinement

Exponentially fast convergence to density operators defined on the codespace $\mathcal{H}_{\text{cat}} = \text{span}\{|0_L\rangle, |1_L\rangle\}$ through dissipative dynamics:

$$\frac{d\rho_B}{dt} = \kappa_2 \mathcal{D}[b^2 - \alpha^2]\rho_B \qquad \bar{\rho}_B^\infty = \sum_{p,q=\pm} c_{pq} |\mathcal{C}_\alpha^q\rangle\langle\mathcal{C}_\alpha^p|$$

Stabilization through dissipation engineering (buffer mode)

Engineering the cat-qubit confinement

Exponentially fast convergence to density operators defined on the codespace $\mathcal{H}_{\text{cat}} = \text{span}\{|0_L\rangle, |1_L\rangle\}$ through dissipative dynamics:

$$\frac{d\rho_B}{dt} = \kappa_2 \mathcal{D}[b^2 - \alpha^2]\rho_B \qquad \bar{\rho}_B^{\infty} = \sum_{p,q=\pm} c_{pq} |\mathcal{C}_\alpha^q\rangle\langle\mathcal{C}_\alpha^p|$$

Stabilization through dissipation engineering (buffer mode)

$$\rho_{AB} \in \mathcal{T}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}) \rightarrow \rho_B \in \mathcal{T}(\mathcal{H}_\mathcal{B})$$

Engineering the cat-qubit confinement

Exponentially fast convergence to density operators defined on the codespace $\mathcal{H}_{\text{cat}} = \text{span}\{|0_L\rangle, |1_L\rangle\}$ through dissipative dynamics:

$$\frac{d\rho_B}{dt} = \kappa_2 \mathcal{D}[b^2 - \alpha^2]\rho_B \qquad \bar{\rho}_B^{\infty} = \sum_{p,q=\pm} c_{pq} |\mathcal{C}_\alpha^q\rangle\langle\mathcal{C}_\alpha^p|$$

Stabilization through dissipation engineering (buffer mode)

$$\rho_{AB} \in \mathcal{T}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}) \to \rho_B \in \mathcal{T}(\mathcal{H}_\mathcal{B})$$

$$\frac{d\rho_{AB}}{dt} = -ig[a^\dagger \otimes (b^2 - \alpha^2) + \text{h.c.}, \rho_{AB}] + \kappa \mathcal{D}[a]\rho_{AB}$$

$$g/\kappa \ll 1 \qquad \to \qquad \frac{d\rho_B}{dt} \sim \frac{4g^2}{\kappa}\mathcal{D}[b^2 - \alpha^2]\rho_B$$

M. Mirrahimi *et al.*, New J. Phys 2014

Outline

- Goal: bridging the gap between two model reduction methods

1. Geometric approach to adiabatic elimination (AE)
2. Time convolutionless (TCL) master equation

Outline

- Goal: bridging the gap between two model reduction methods

1. Geometric approach to adiabatic elimination (AE)
2. Time convolutionless (TCL) master equation

- Result: TCL approach to AE

Outline

- Goal: bridging the gap between two model reduction methods

1. Geometric approach to adiabatic elimination (AE)

2. Time convolutionless (TCL) master equation

- Result: TCL approach to AE

- Example: retrieving the cat qubit confinement

Outline

- Goal: bridging the gap between two model reduction methods

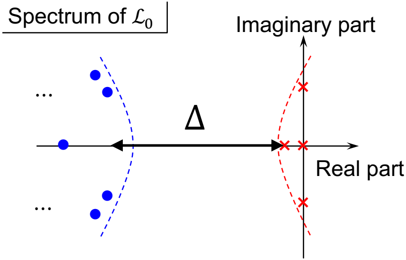
1. Geometric approach to adiabatic elimination (AE)
2. Time convolutionless (TCL) master equation

- Result: TCL approach to AE
- Example: retrieving the cat qubit confinement
- Conclusion: discussion and outlook

1. Adiabatic elimination

$$\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t) = (\mathcal{L}_0 + \epsilon\mathcal{L}_1)\rho(t) \qquad \rho(t) \in T(\mathcal{H})$$

- The **spectrum** of $\mathcal{L}_0$ is **gapped** on the real axis



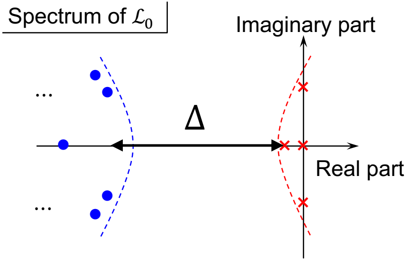
$$(\mathcal{L}_0 - \lambda_i)\,|X_i^R\rangle\rangle = 0$$

$$\langle\langle X_i^L|\,(\mathcal{L}_0 - \lambda_i) = 0$$

1. Adiabatic elimination

$$\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t) = (\mathcal{L}_0 + \epsilon\mathcal{L}_1)\rho(t) \qquad \rho(t) \in T(\mathcal{H})$$

- The **spectrum** of $\mathcal{L}_0$ is **gapped** on the real axis



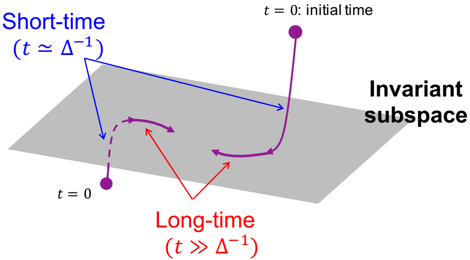
$$(\mathcal{L}_0 - \lambda_i)\,|X_i^R\rangle\!\rangle = 0$$

$$\langle\!\langle X_i^L|\,(\mathcal{L}_0 - \lambda_i) = 0$$

- For $0 < \epsilon|\mathcal{L}_1| \ll \Delta$ the spectrum of $\mathcal{L}$ is **still gapped**

1. Adiabatic elimination

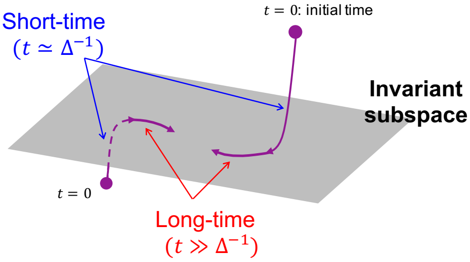
$$\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t) = (\mathcal{L}_0 + \epsilon\mathcal{L}_1)\rho(t) \qquad \rho(t) \in T(\mathcal{H})$$



Goal: reduced model $\vec{x}(t)$ of the dynamics on the invariant subspace.

1. Adiabatic elimination

$$\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t) = (\mathcal{L}_0 + \epsilon\mathcal{L}_1)\rho(t) \qquad \rho(t) \in T(\mathcal{H})$$



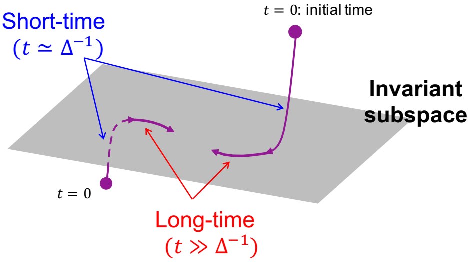
Goal: reduced model $\vec{x}(t)$ of the dynamics on the invariant subspace.

$$\frac{d}{dt}\vec{x}(t) = \mathcal{F}^{(\epsilon)}\vec{x}(t)$$

$$\rho(t) = \mathcal{K}^{(\epsilon)}\vec{x}(t)$$

1. Adiabatic elimination

$$\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t) = (\mathcal{L}_0 + \epsilon\mathcal{L}_1)\rho(t) \qquad \rho(t) \in T(\mathcal{H})$$



Goal: reduced model $\vec{x}(t)$ of the dynamics on the invariant subspace.

$$\frac{d}{dt}\vec{x}(t) = \mathcal{F}^{(\epsilon)}\vec{x}(t)$$

$$\rho(t) = \mathcal{K}^{(\epsilon)}\vec{x}(t)$$

Invariance condition: $\mathcal{K}^{(\epsilon)}\mathcal{F}^{(\epsilon)} = \mathcal{L}\mathcal{K}^{(\epsilon)}$

Rémi Azouit *et al.* 2016 IEEE 55th CDC

2. Time-convolutionless master equation

$$\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t) = (\mathcal{L}_0 + \epsilon\mathcal{L}_1)\rho(t) \qquad \rho(t) \in T(\mathcal{H})$$

2. Time-convolutionless master equation

$$\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t) = (\mathcal{L}_0 + \epsilon\mathcal{L}_1)\rho(t) \qquad \rho(t) \in T(\mathcal{H})$$

Projected dynamics $\mathcal{P}^2 = \mathcal{P}$, $\mathcal{Q} = \mathcal{I} - \mathcal{P}$, $\mathcal{QP} = \mathcal{PQ} = 0$.

$$\frac{d}{dt}\mathcal{P}\rho(t) = \mathcal{PL}(\mathcal{P} + \mathcal{Q})\rho(t) \qquad \frac{d}{dt}\mathcal{Q}\rho(t) = \mathcal{QL}(\mathcal{P} + \mathcal{Q})\rho(t)$$

2. Time-convolutionless master equation

$$\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t) = (\mathcal{L}_0 + \epsilon\mathcal{L}_1)\rho(t) \qquad \rho(t) \in T(\mathcal{H})$$

Projected dynamics $\mathcal{P}^2 = \mathcal{P}$, $\mathcal{Q} = \mathcal{I} - \mathcal{P}$, $\mathcal{Q}\mathcal{P} = \mathcal{P}\mathcal{Q} = 0$.

$$\frac{d}{dt}\color{red}{\mathcal{P}\rho(t)} = \mathcal{P}\mathcal{L}(\mathcal{P} + \mathcal{Q})\rho(t) \qquad \frac{d}{dt}\color{blue}{\mathcal{Q}\rho(t)} = \mathcal{Q}\mathcal{L}(\mathcal{P} + \mathcal{Q})\rho(t)$$

$$\rho(t) = \color{red}{\mathcal{P}\rho(t)} + \overbrace{\color{blue}{\mathcal{Q}\rho(t)}}^{\text{formal solution}} \quad \rightarrow \quad \text{TCL master eq } (\textbf{exact, time local})$$

2. Time-convolutionless master equation

$$\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t) = (\mathcal{L}_0 + \epsilon\mathcal{L}_1)\rho(t) \qquad \rho(t) \in T(\mathcal{H})$$

Projected dynamics $\mathcal{P}^2 = \mathcal{P}$, $\mathcal{Q} = \mathcal{I} - \mathcal{P}$, $\mathcal{QP} = \mathcal{PQ} = 0$.

$$\frac{d}{dt}\mathcal{P}\rho(t) = \mathcal{PL}(\mathcal{P} + \mathcal{Q})\rho(t) \qquad \frac{d}{dt}\mathcal{Q}\rho(t) = \mathcal{QL}(\mathcal{P} + \mathcal{Q})\rho(t)$$

$$\rho(t) = \mathcal{P}\rho(t) + \overbrace{\mathcal{Q}\rho(t)}^{\text{formal solution}} \quad \to \quad \text{TCL master eq } (\textbf{exact, time local})$$

$$\mathcal{P} = \sum_s |X_s^R\rangle\!\rangle\langle\!\langle X_s^L| \qquad \mathcal{Q}\rho(0) = 0$$

Breuer & Petruccione, Open quantum systems theory (2004).

Projective formulation of adiabatic elimination

$$\rho(t) = [1 - \Sigma(t)]^{-1}\mathcal{P}\rho(t)$$

$$\frac{d}{dt}\mathcal{P}\rho(t) = \mathcal{P}\mathcal{L}[1 - \Sigma(t)]^{-1}\mathcal{P}\rho(t)$$

Projective formulation of adiabatic elimination

$$\rho(t) = [1 - \Sigma(t)]^{-1} \mathcal{P}\rho(t)$$
$$\frac{d}{dt}\mathcal{P}\rho(t) = \mathcal{P}\mathcal{L}[1 - \Sigma(t)]^{-1}\mathcal{P}\rho(t)$$

For simplicity, we consider $\Sigma(t)$ up to first order:

$$[\mathcal{P}, \mathcal{L}_0] = 0 \quad \rightarrow \quad \Sigma_1(t) = \epsilon \int_0^t d\tau \, e^{\mathcal{L}_0 \tau} \mathcal{Q}\mathcal{L}_1 e^{-\mathcal{L}_0 \tau} \mathcal{P} + \mathcal{O}(\epsilon^2)$$

Projective formulation of adiabatic elimination

$$\rho(t) = \big[1 + \Sigma_1(t)\big]\mathcal{P}\rho(t) + \mathcal{O}(\epsilon^2)$$

$$\frac{d}{dt}\mathcal{P}\rho(t) = \mathcal{P}\mathcal{L}\big[1 + \Sigma_1(t)\big]\mathcal{P}\rho(t) + \mathcal{O}(\epsilon^3)$$

For simplicity, we consider $\Sigma(t)$ up to first order:

$$[\mathcal{P}, \mathcal{L}_0] = 0 \quad \rightarrow \quad \Sigma_1(t) = \epsilon \int_0^t d\tau \, e^{\mathcal{L}_0 \tau} \mathcal{Q}\mathcal{L}_1 e^{-\mathcal{L}_0 \tau}\mathcal{P} + \mathcal{O}(\epsilon^2)$$

Projective formulation of adiabatic elimination

$$\rho(t) = \left[1 + \Sigma_1(t)\right]\mathcal{P}\rho(t) + \mathcal{O}(\epsilon^2)$$

$$\frac{d}{dt}\mathcal{P}\rho(t) = \mathcal{P}\mathcal{L}\left[1 + \Sigma_1(t)\right]\mathcal{P}\rho(t) + \mathcal{O}(\epsilon^3)$$

For simplicity, we consider $\Sigma(t)$ up to first order:

$$[\mathcal{P}, \mathcal{L}_0] = 0 \quad \rightarrow \quad \Sigma_1(t) = \epsilon \int_0^t d\tau\, e^{\mathcal{L}_0 \tau} \mathcal{Q}\mathcal{L}_1 e^{-\mathcal{L}_0 \tau}\mathcal{P} + \mathcal{O}(\epsilon^2)$$

After the fast relaxation phase, $t \gg \Delta^{-1}$, we expect

$$\Sigma_1(t) \approx \Sigma_1 = lim_{t\rightarrow\infty}\Sigma_1(t)$$

Projective formulation of adiabatic elimination

$$\rho(t) = \left[1 + \Sigma_1(t)\right]\mathcal{P}\rho(t) + \mathcal{O}(\epsilon^2)$$

$$\frac{d}{dt}\mathcal{P}\rho(t) = \mathcal{P}\mathcal{L}\left[1 + \Sigma_1(t)\right]\mathcal{P}\rho(t) + \mathcal{O}(\epsilon^3)$$

For simplicity, we consider $\Sigma(t)$ up to first order:

$$[\mathcal{P}, \mathcal{L}_0] = 0 \quad \rightarrow \quad \Sigma_1 = \epsilon \int_0^{+\infty} d\tau \, e^{\mathcal{L}_0\tau}\mathcal{Q}\mathcal{L}_1 e^{-\mathcal{L}_0\tau}\mathcal{P} + \mathcal{O}(\epsilon^2)$$

After the fast relaxation phase, $t \gg \Delta^{-1}$, we expect

$$\Sigma_1(t) \approx \Sigma_1 = lim_{t\rightarrow\infty}\Sigma_1(t)$$

Showing the equivalence of AE and TCL

$$\rho(t) = \left[1 + \Sigma_1\right] \mathcal{P}\rho(t) + \mathcal{O}(\epsilon^2)$$

$$\frac{d}{dt}\mathcal{P}\rho(t) = \mathcal{P}\mathcal{L}\left[1 + \Sigma_1\right]\mathcal{P}\rho(t) + \mathcal{O}(\epsilon^3)$$

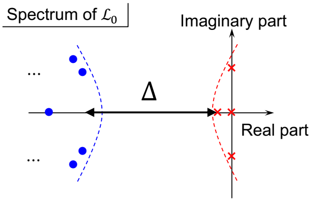
Showing the equivalence of AE and TCL

$$\rho(t) = \left[1 + \Sigma_1\right]\mathcal{P}\rho(t) + \mathcal{O}(\epsilon^2)$$

$$\frac{d}{dt}\mathcal{P}\rho(t) = \mathcal{P}\mathcal{L}\left[1 + \Sigma_1\right]\mathcal{P}\rho(t) + \mathcal{O}(\epsilon^3)$$

We parametrize the reduced dynamics $\vec{x}(t)$ on the invariant subspace of the **slow modes** $\mathcal{P}\,|\rho(t)\rangle\!\rangle = \sum_s x_s(t)\,|X_s^R\rangle\!\rangle$:

$$x_s(t) = \langle\!\langle X_s^L|\rho(t)\rangle\!\rangle$$



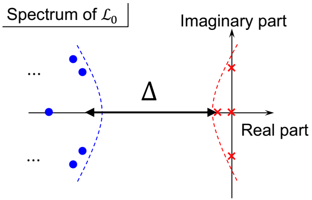
Showing the equivalence of AE and TCL

$$\rho(t) = \left[1 + \Sigma_1\right] \chi_R \vec{x}(t) + \mathcal{O}(\epsilon^2)$$

$$\frac{d}{dt}\vec{x}(t) = \chi_L^\dagger \mathcal{L} \left[1 + \Sigma_1\right] \chi_R \vec{x}(t) + \mathcal{O}(\epsilon^3)$$

We parametrize the reduced dynamics $\vec{x}(t)$ on the invariant subspace of the **slow modes** $\mathcal{P}\,|\rho(t)\rangle\rangle = \sum_s x_s(t)\,|X_s^R\rangle\rangle$:

$$x_s(t) = \langle\langle X_s^L | \rho(t) \rangle\rangle$$



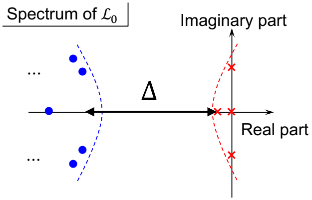
Showing the equivalence of AE and TCL

$$\rho(t) = \left[1 + \Sigma_1\right]\chi_R \vec{x}(t) \quad \rightarrow \quad \rho(t) = \mathcal{K}^{(\epsilon)}_{\mathsf{TCL}}\, \vec{x}(t)$$

$$\frac{d}{dt}\vec{x}(t) = \chi_L^\dagger \mathcal{L}\left[1 + \Sigma_1\right]\chi_R \vec{x}(t) \quad \rightarrow \quad \frac{d}{dt}\vec{x}(t) = \mathcal{F}^{(\epsilon)}_{\mathsf{TCL}}\, \vec{x}(t)$$

We parametrize the reduced dynamics $\vec{x}(t)$ on the invariant subspace of the **slow modes** $\mathcal{P}\,|\rho(t)\rangle\rangle = \sum_s x_s(t)\,|X_s^R\rangle\rangle$:

$$x_s(t) = \langle\langle X_s^L|\rho(t)\rangle\rangle$$



TCL version of $\mathcal{K}$ and $\mathcal{F}$

$$\rho(t) = \left[1 + \Sigma_1\right]\mathcal{P}\rho(t) + \mathcal{O}(\epsilon^2)$$
$$\frac{d}{dt}\mathcal{P}\rho(t) = \mathcal{P}\mathcal{L}\left[1 + \Sigma_1\right]\mathcal{P}\rho(t) + \mathcal{O}(\epsilon^3)$$

We define χ_R and $\chi_L^\dagger$ such that $\mathcal{P} = \chi_R\chi_L^\dagger$, $\chi_L^\dagger\chi_R = 1$:
$\chi_R = \left[|X^R_{s=1}\rangle\rangle \; |X^R_{s=2}\rangle\rangle \; \cdots\right] \quad \chi_L^\dagger = \left[|X^L_{s=1}\rangle\rangle \; |X^L_{s=2}\rangle\rangle \; \cdots\right]^\dagger$

TCL version of $\mathcal{K}$ and $\mathcal{F}$

$$\rho(t) = \left[1 + \Sigma\right] \chi_R \chi_L^\dagger \rho(t) + \mathcal{O}(\epsilon^2)$$

$$\frac{d}{dt} \chi_R \chi_L^\dagger \rho(t) = \chi_R \chi_L^\dagger \mathcal{L} \left[1 + \Sigma\right] \chi_R \chi_L^\dagger \rho(t) + \mathcal{O}(\epsilon^3)$$

We define χ_R and $\chi_L^\dagger$ such that $\mathcal{P} = \chi_R \chi_L^\dagger$, $\chi_L^\dagger \chi_R = 1$:

$\chi_R = \left[|X^R_{s=1}\rangle\rangle \; |X^R_{s=2}\rangle\rangle \; \cdots \right] \quad \chi_L^\dagger = \left[|X^L_{s=1}\rangle\rangle \; |X^L_{s=2}\rangle\rangle \; \cdots \right]^\dagger$

TCL version of $\mathcal{K}$ and $\mathcal{F}$

$$\rho(t) = \left[1 + \Sigma\right] \chi_R \chi_L^\dagger \rho(t) + \mathcal{O}(\epsilon^2)$$
$$\frac{d}{dt} \chi_R \chi_L^\dagger \rho(t) = \chi_R \chi_L^\dagger \mathcal{L} \left[1 + \Sigma\right] \chi_R \chi_L^\dagger \rho(t) + \mathcal{O}(\epsilon^3)$$

We define χ_R and $\chi_L^\dagger$ such that $\mathcal{P} = \chi_R \chi_L^\dagger$, $\chi_L^\dagger \chi_R = 1$:
$\chi_R = \left[|X_{s=1}^R\rangle\rangle \; |X_{s=2}^R\rangle\rangle \; \cdots \right] \quad \chi_L^\dagger = \left[|X_{s=1}^L\rangle\rangle \; |X_{s=2}^L\rangle\rangle \; \cdots \right]^\dagger$

We choose a parametrization $\vec{x}(t) = \chi_L^\dagger \, |\rho(t)\rangle\rangle$.

TCL version of $\mathcal{K}$ and $\mathcal{F}$

$$\rho(t) = \left[1 + \Sigma\right]\chi_R \chi_L^\dagger \rho(t) + \mathcal{O}(\epsilon^2)$$
$$\frac{d}{dt}\chi_R \chi_L^\dagger \rho(t) = \chi_R \chi_L^\dagger \mathcal{L}\left[1 + \Sigma\right]\chi_R \chi_L^\dagger \rho(t) + \mathcal{O}(\epsilon^3)$$

We define χ_R and $\chi_L^\dagger$ such that $\mathcal{P} = \chi_R \chi_L^\dagger$, $\chi_L^\dagger \chi_R = 1$:
$\chi_R = \left[|X^R_{s=1}\rangle\!\rangle \; |X^R_{s=2}\rangle\!\rangle \; \cdots\right] \quad \chi_L^\dagger = \left[|X^L_{s=1}\rangle\!\rangle \; |X^L_{s=2}\rangle\!\rangle \; \cdots\right]^\dagger$

We choose a parametrization $\vec{x}(t) = \chi_L^\dagger \, |\rho(t)\rangle\!\rangle$.

TCL version of $\mathcal{K}$ and $\mathcal{F}$

$$\rho(t) = \left[1 + \Sigma\right]\chi_R \vec{x}(t) + \mathcal{O}(\epsilon^2)$$
$$\frac{d}{dt}\vec{x}(t) = \chi_L^\dagger \mathcal{L}\left[1 + \Sigma\right]\chi_R \vec{x}(t) + \mathcal{O}(\epsilon^3)$$

We define χ_R and $\chi_L^\dagger$ such that $\mathcal{P} = \chi_R \chi_L^\dagger$, $\chi_L^\dagger \chi_R = 1$:
$\chi_R = \left[|X^R_{s=1}\rangle\rangle \; |X^R_{s=2}\rangle\rangle \; \cdots\right] \quad \chi_L^\dagger = \left[|X^L_{s=1}\rangle\rangle \; |X^L_{s=2}\rangle\rangle \; \cdots\right]^\dagger$

We choose a parametrization $\vec{x}(t) = \chi_L^\dagger \, |\rho(t)\rangle\rangle$.

TCL version of $\mathcal{K}$ and $\mathcal{F}$

$$\rho(t) = \left[1 + \Sigma\right]\chi_R \vec{x}(t) + \mathcal{O}(\epsilon^2)$$
$$\frac{d}{dt}\vec{x}(t) = \chi_L^\dagger \mathcal{L}\left[1 + \Sigma\right]\chi_R \vec{x}(t) + \mathcal{O}(\epsilon^3)$$

We define χ_R and $\chi_L^\dagger$ such that $\mathcal{P} = \chi_R \chi_L^\dagger$, $\chi_L^\dagger \chi_R = 1$:
$\chi_R = \left[|X^R_{s=1}\rangle\!\rangle \; |X^R_{s=2}\rangle\!\rangle \; \cdots\right] \quad \chi_L^\dagger = \left[|X^L_{s=1}\rangle\!\rangle \; |X^L_{s=2}\rangle\!\rangle \; \cdots\right]^\dagger$

We choose a parametrization $\vec{x}(t) = \chi_L^\dagger \, |\rho(t)\rangle\!\rangle$.

$\mathcal{K}^{(\epsilon)}_{\mathrm{TCL}} = \left[1 + \Sigma\right]\chi_R \quad \mathcal{F}^{(\epsilon)}_{\mathrm{TCL}} = \chi_L^\dagger \mathcal{L}\left[1 + \Sigma\right]\chi_R$ correspond to AE!

Main result

Theorem. $\mathcal{K}^{(\epsilon)}_{\mathsf{TCL}}$ and $\mathcal{F}^{(\epsilon)}_{\mathsf{TCL}}$ satisfy the invariance condition:

$$\mathcal{K}^{(\epsilon)}_{\mathsf{TCL}}\mathcal{F}^{(\epsilon)}_{\mathsf{TCL}} = \mathcal{L}\mathcal{K}^{(\epsilon)}_{\mathsf{TCL}} + \mathcal{O}(\epsilon^2)$$

Main result

Theorem. $\mathcal{K}^{(\epsilon)}_{\mathsf{TCL}}$ and $\mathcal{F}^{(\epsilon)}_{\mathsf{TCL}}$ satisfy the invariance condition:

$$\mathcal{K}^{(\epsilon)}_{\mathsf{TCL}}\mathcal{F}^{(\epsilon)}_{\mathsf{TCL}} = \mathcal{L}\mathcal{K}^{(\epsilon)}_{\mathsf{TCL}} + \mathcal{O}(\epsilon^2)$$

At long times $t \gg \Delta^{-1}$, the dynamics are constrained on the invariant subspace with $\epsilon > 0$, **preserved by $\mathcal{L}$**.

Main result

Theorem. $\mathcal{K}^{(\epsilon)}_{\mathsf{TCL}}$ and $\mathcal{F}^{(\epsilon)}_{\mathsf{TCL}}$ satisfy the invariance condition:

$$\mathcal{K}^{(\epsilon)}_{\mathsf{TCL}}\mathcal{F}^{(\epsilon)}_{\mathsf{TCL}} = \mathcal{L}\mathcal{K}^{(\epsilon)}_{\mathsf{TCL}}$$

At long times $t \gg \Delta^{-1}$, the dynamics are constrained on the invariant subspace with $\epsilon > 0$, **preserved by $\mathcal{L}$**.

With similar arguments we can treat higher order terms:

Main result

Theorem. $\mathcal{K}^{(\epsilon)}_{\mathsf{TCL}}$ and $\mathcal{F}^{(\epsilon)}_{\mathsf{TCL}}$ satisfy the invariance condition:

$$\mathcal{K}^{(\epsilon)}_{\mathsf{TCL}}\mathcal{F}^{(\epsilon)}_{\mathsf{TCL}} = \mathcal{L}\mathcal{K}^{(\epsilon)}_{\mathsf{TCL}}$$

At long times $t \gg \Delta^{-1}$, the dynamics are constrained on the invariant subspace with $\epsilon > 0$, **preserved by $\mathcal{L}$**.

With similar arguments we can treat higher order terms:

The theorem is proven up to **infinite order in** ϵ.

Example: bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Flowchart

- Identifying $\mathcal{L}_0$ and $\mathcal{L}_1$
- Identifying the surviving modes
- Specifying the definition of the projector $\mathcal{P}$

Example: bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Flowchart

- Identifying $\mathcal{L}_0$ and $\mathcal{L}_1$
- Identifying the surviving modes
- Specifying the definition of the projector $\mathcal{P}$

$$\frac{d\rho}{dt} = \mathcal{L}\rho \qquad \mathcal{L} = \underbrace{\mathcal{L}_A \otimes \mathcal{I}_B}_{\mathcal{L}_0} + \underbrace{\mathcal{L}_B + \mathcal{L}_{\text{int}}}_{\epsilon\mathcal{L}_1}$$

$\mathcal{L}_A \bullet = -i\omega[a^\dagger a, \bullet] + \kappa\mathcal{D}[a]\bullet$ $\quad$ $\mathcal{H}_A$ damped harmonic oscillator

$\mathcal{L}_B \bullet$ $\quad$ unspecified $\quad$ $\mathcal{H}_B$ arbitrary system $\dim \mathcal{H}_B = N$

$\mathcal{L}_{\text{int}} \bullet = -ig[a^\dagger \otimes L + a \otimes L^\dagger, \bullet]$ $\quad$ $|\mathcal{L}_B|, g \ll \kappa$

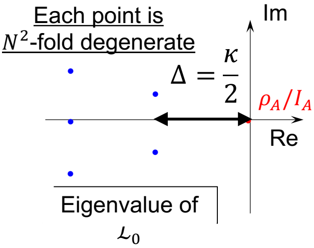
Example: bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Flowchart

- Identifying $\mathcal{L}_0$ and $\mathcal{L}_1$
- Identifying the surviving modes
- Specifying the definition of the projector $\mathcal{P}$

$\mathcal{L}_0 = \mathcal{L}_A \otimes \mathcal{I}_B$ is trivial on B.

$\mathcal{L}_A \, \|0\rangle\langle 0|\rangle\rangle = 0 \quad \langle\langle I_A | \, \mathcal{L}_A = 0$



Example: bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Flowchart

- Identifying $\mathcal{L}_0$ and $\mathcal{L}_1$
- Identifying the surviving modes
- Specifying the definition of the projector $\mathcal{P}$

$$X^R_{s=m,n} = \bar{\rho}_A \otimes |b_m\rangle\langle b_n| \quad X^L_{s=m,n} = I_A \otimes |b_m\rangle\langle b_n| \quad \langle\langle X^L_s | X^R_r \rangle\rangle = \delta_{sr}$$

$$\mathcal{P}\rho = \sum_{m,n=1}^{N} \langle\langle X^L_{s=m,n} | \rho \rangle\rangle X^R_{s=m,n} = \sum_{m,n=1}^{N} \bar{\rho}_A \otimes \underbrace{\Pi_m}_{|b_m\rangle\langle b_m|} \operatorname{tr}_A(\rho)\Pi_n = \bar{\rho}_A \otimes \operatorname{tr}_A(\rho)$$

$$\mathcal{P}\rho = |0\rangle\langle 0| \otimes \underbrace{\operatorname{tr}_A(\rho)}_{\rho_B}$$

ρ_B parametrizes the invariant subspace.

Example: bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

We remark: $\mathcal{P}\mathcal{L}_0\rho = |0\rangle\langle 0| \otimes \mathrm{tr}_A(\mathcal{L}_A \otimes \mathcal{I}_B\rho) = 0 \implies \mathcal{P}\mathcal{L}_0 = 0$

$$\frac{d}{dt}\mathcal{P}\rho(t) = |0\rangle\langle 0| \otimes \frac{d}{dt}\rho_B(t)$$

$$= \left(\epsilon\mathcal{P}\mathcal{L}_1 + \epsilon^2 \int_0^{+\infty} ds\, e^{\mathcal{L}_0 s}\mathcal{Q}\mathcal{L}_1\right)|0\rangle\langle 0| \otimes \rho_B(t) + \mathcal{O}(\epsilon^3)$$

Example: bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

We remark: $\mathcal{P}\mathcal{L}_0\rho = |0\rangle\langle 0| \otimes \mathrm{tr}_A(\mathcal{L}_A \otimes \mathcal{I}_B\rho) = 0 \implies \mathcal{P}\mathcal{L}_0 = 0$

$$\frac{d}{dt}\mathcal{P}\rho(t) = |0\rangle\langle 0| \otimes \frac{d}{dt}\rho_B(t)$$
$$= \left(\epsilon\mathcal{P}\mathcal{L}_1 + \epsilon^2 \int_0^{+\infty} ds\, e^{\mathcal{L}_0 s}\mathcal{Q}\mathcal{L}_1\right) |0\rangle\langle 0| \otimes \rho_B(t) + \mathcal{O}(\epsilon^3)$$

Reduced model of the dynamics on B

$$\frac{d}{dt}\rho_B(t) = \mathcal{L}_B\rho_B(t) - i\frac{4g^2\omega}{\kappa^2 + 4\omega^2}[L^\dagger L, \rho_B(t)] + \frac{4g^2}{\kappa + 4\omega^2/\kappa}\mathcal{D}[L]\rho_B(t)$$

Example: bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

We remark: $\mathcal{P}\mathcal{L}_0\rho = |0\rangle\langle 0| \otimes \mathrm{tr}_A(\mathcal{L}_A \otimes \mathcal{I}_B \rho) = 0 \implies \mathcal{P}\mathcal{L}_0 = 0$

$$\frac{d}{dt}\mathcal{P}\rho(t) = |0\rangle\langle 0| \otimes \frac{d}{dt}\rho_B(t)$$
$$= \left(\epsilon\mathcal{P}\mathcal{L}_1 + \epsilon^2 \int_0^{+\infty} ds\, e^{\mathcal{L}_0 s}\mathcal{Q}\mathcal{L}_1\right)|0\rangle\langle 0| \otimes \rho_B(t) + \mathcal{O}(\epsilon^3)$$

Reduced model of the dynamics on B

$$\frac{d}{dt}\rho_B(t) = \mathcal{L}_B\rho_B(t) - i\frac{4g^2\omega}{\kappa^2 + 4\omega^2}[L^\dagger L, \rho_B(t)] + \frac{4g^2}{\kappa + 4\omega^2/\kappa}\mathcal{D}[L]\rho_B(t)$$

Hamiltonian correction and inherited dissipation from A.

Summary & Outlook

TCL master equations give alternative formulation of AE.

Summary & Outlook

TCL master equations give alternative formulation of AE.

- TCL gains a geometric interpretation (reduced description of the total dyanmics based on time-scale separation)
- AE gains a systematic way to make calculations

Summary & Outlook

TCL master equations give alternative formulation of AE.

- TCL gains a geometric interpretation (reduced description of the total dyanmics based on time-scale separation)
- AE gains a systematic way to make calculations

Application to any system with time-scale separation (c-QED, dynamic nuclear polarization [A. Karabanov *et al.*, PRL (2015)])

Summary & Outlook

TCL master equations give alternative formulation of AE.

- TCL gains a geometric interpretation (reduced description of the total dyanmics based on time-scale separation)
- AE gains a systematic way to make calculations

Application to any system with time-scale separation (c-QED, dynamic nuclear polarization [A. Karabanov *et al.*, PRL (2015)])

In prep. with Masaaki Tokieda

Time-Convolutionless Master Equation Applied to Adiabatic Elimination

Backup slides

Sketch of the theorem's proof (1)

$$\mathcal{P}(t) = [1 - \Sigma(t)]^{-1}\mathcal{P} \qquad \Sigma(t) = \int_0^t d\tau\, e^{\mathcal{QLQ}\tau}\mathcal{QLP}e^{-\mathcal{L}\tau}$$

Sketch of the theorem's proof (1)

$$\mathcal{P}(t) = [1 - \Sigma(t)]^{-1}\mathcal{P} \qquad \Sigma(t) = \int_0^t d\tau\, e^{\mathcal{QLQ}\tau}\mathcal{QLP}e^{-\mathcal{L}\tau}$$

1. Rewrite $\mathcal{P}(t)$ in terms of $e^{\mathcal{L}t}$ [C. Timm, Phys. Rev. B (2008)]

$$e^{\mathcal{L}t}\mathcal{P}[\mathcal{Q} + \mathcal{P}e^{\mathcal{L}t}\mathcal{P}]^{-1}\mathcal{P}$$

2. Express $e^{\mathcal{L}t}$ in the eigenbasis of $\mathcal{L}$

Sketch of the theorem's proof (1)

$$\mathcal{P}(t) = [1 - \Sigma(t)]^{-1}\mathcal{P} \qquad \Sigma(t) = \int_0^t d\tau\, e^{\mathcal{QLQ}\tau}\mathcal{QLP}e^{-\mathcal{L}\tau}$$

1. Rewrite $\mathcal{P}(t)$ in terms of $e^{\mathcal{L}t}$ [C. Timm, Phys. Rev. B (2008)]

$$e^{\mathcal{L}t}\mathcal{P}[\mathcal{Q} + \mathcal{P}e^{\mathcal{L}t}\mathcal{P}]^{-1}\mathcal{P}$$

2. Express $e^{\mathcal{L}t}$ in the eigenbasis of $\mathcal{L}$
3. Show the existence of the **projection** $\mathcal{P}(\infty)$

Sketch of the theorem's proof (1)

$$\mathcal{P}(t) = [1 - \Sigma(t)]^{-1}\mathcal{P} \qquad \Sigma(t) = \int_0^t d\tau \, e^{\mathcal{QLQ}\tau}\mathcal{QLP}e^{-\mathcal{L}\tau}$$

1. Rewrite $\mathcal{P}(t)$ in terms of $e^{\mathcal{L}t}$ [C. Timm, Phys. Rev. B (2008)]

$$e^{\mathcal{L}t}\mathcal{P}[\mathcal{Q} + \mathcal{P}e^{\mathcal{L}t}\mathcal{P}]^{-1}\mathcal{P}$$

2. Express $e^{\mathcal{L}t}$ in the eigenbasis of $\mathcal{L}$
3. Show the existence of the **projection** $\mathcal{P}(\infty)$
   - Image: invariant subspace $\epsilon > 0$

Sketch of the theorem's proof (1)

$$\mathcal{P}(t) = [1 - \Sigma(t)]^{-1}\mathcal{P} \qquad \Sigma(t) = \int_0^t d\tau\, e^{\mathcal{QLQ}\tau}\mathcal{QLP}e^{-\mathcal{L}\tau}$$

1. Rewrite $\mathcal{P}(t)$ in terms of $e^{\mathcal{L}t}$ [C. Timm, Phys. Rev. B (2008)]

$$e^{\mathcal{L}t}\mathcal{P}[\mathcal{Q} + \mathcal{P}e^{\mathcal{L}t}\mathcal{P}]^{-1}\mathcal{P}$$

2. Express $e^{\mathcal{L}t}$ in the eigenbasis of $\mathcal{L}$
3. Show the existence of the **projection** $\mathcal{P}(\infty)$
    - Image: invariant subspace $\epsilon > 0$
    - Kernel: subspace spanned by the fast relaxing modes $\epsilon = 0$

Sketch of the theorem's proof (1)

$$\mathcal{P}(t) = [1 - \Sigma(t)]^{-1}\mathcal{P} \qquad \Sigma(t) = \int_0^t d\tau\, e^{\mathcal{QLQ}\tau}\mathcal{QLP}e^{-\mathcal{L}\tau}$$

1. Rewrite $\mathcal{P}(t)$ in terms of $e^{\mathcal{L}t}$ [C. Timm, Phys. Rev. B (2008)]

$$e^{\mathcal{L}t}\mathcal{P}[\mathcal{Q} + \mathcal{P}e^{\mathcal{L}t}\mathcal{P}]^{-1}\mathcal{P}$$

2. Express $e^{\mathcal{L}t}$ in the eigenbasis of $\mathcal{L}$
3. Show the existence of the **projection** $\mathcal{P}(\infty)$
   - Image: invariant subspace $\epsilon > 0$
   - Kernel: subspace spanned by the fast relaxing modes $\epsilon = 0$
4. With $\Delta^{(\epsilon)} \approx \Delta$, show that

$$\mathcal{P}(t) = \mathcal{P}(\infty) + (\mathcal{P}(t) - \mathcal{P}(\infty)), \quad \mathcal{P}(t) - \mathcal{P}(\infty) \propto e^{-\Delta t}$$

Sketch of the theorem's proof (2)

$\mathcal{P}(\infty)\rho(t)$ projects a state to the invariant subspace with $\epsilon > 0$ (invariant w.r.t. $\mathcal{L}$). It follows:

$$\mathcal{Q}(\infty)\mathcal{L}(\infty)\mathcal{P}(\infty) = 0$$

Sketch of the theorem's proof (2)

$\mathcal{P}(\infty)\rho(t)$ projects a state to the invariant subspace with $\epsilon > 0$ (invariant w.r.t. $\mathcal{L}$). It follows:

$$\mathcal{Q}(\infty)\mathcal{L}(\infty)\mathcal{P}(\infty) = 0$$

By definition $\mathcal{Q}(\infty) = 1 - \mathcal{P}(\infty)$

$$\implies \mathcal{P}(\infty)\mathcal{L}\mathcal{P}(\infty) = \mathcal{L}\mathcal{P}(\infty)$$

Sketch of the theorem's proof (2)

$\mathcal{P}(\infty)\rho(t)$ projects a state to the invariant subspace with $\epsilon > 0$ (invariant w.r.t. $\mathcal{L}$). It follows:

$$\mathcal{Q}(\infty)\mathcal{L}(\infty)\mathcal{P}(\infty) = 0$$

By definition $\mathcal{Q}(\infty) = 1 - \mathcal{P}(\infty)$

$$\implies \mathcal{P}(\infty)\mathcal{L}\mathcal{P}(\infty) = \mathcal{L}\mathcal{P}(\infty)$$

which is equivalent to the invariance condition:

$$\mathcal{K}_{\mathsf{TCL}}\mathcal{F}_{\mathsf{TCL}} = \mathcal{L}\mathcal{K}_{\mathsf{TCL}}$$

where $\mathcal{K}_{\mathsf{TCL}} = \mathcal{P}(\infty)\chi_R$ and $\mathcal{F}_{\mathsf{TCL}} = \chi_L^\dagger \mathcal{L}\mathcal{P}(\infty)\chi_R$ correspond to the AE maps, and satisfy the invariance condition.

Derivation of TCL equation (higher orders)

From $\rho(t) = \mathcal{P}\rho(t) + \mathcal{Q}\rho(t)$ with the formal solution of $\mathcal{Q}\rho(t)$ we obtain:

$$\rho(t) = \mathcal{J}(t)\mathcal{Q}\rho(0) + [1 - \Sigma(t)]^{-1}\mathcal{P}\rho(t)$$
$$\frac{d}{dt}\mathcal{P}\rho(t) = \mathcal{P}\mathcal{L}\mathcal{J}(t)\mathcal{Q}\rho(0) + \mathcal{P}\mathcal{L}[1 - \Sigma(t)]^{-1}\mathcal{P}\rho(t)$$

where we assumed that $[\mathcal{P}, \mathcal{L}_0] = 0$

$$\mathcal{J}(t) = [1 - \Sigma(t)]^{-1} e^{\mathcal{QLQ}t}\mathcal{Q}$$
$$\Sigma(t) = \epsilon \int_0^t d\tau \, e^{\mathcal{QLQ}\tau}\mathcal{Q}\mathcal{L}_1\mathcal{P}e^{-\mathcal{L}\tau}.$$

We define:

$$\mathcal{P}(t) = [1 - \Sigma(t)]^{-1}\mathcal{P}$$

To be precise on the expansion in ϵ

$$[1-\Sigma(t)]^{-1} = 1 + \epsilon\Sigma_1(t) + \epsilon^2\Sigma_2(t) + \mathcal{O}(\epsilon^3)$$

$$\downarrow$$

$$\begin{aligned}
&\mathcal{P}\mathcal{L}[1-\Sigma(t)]^{-1}\mathcal{P} \\
&= \mathcal{P}\mathcal{L}[1 + \epsilon\Sigma_1(t) + \epsilon^2\Sigma_2(t) + \mathcal{O}(\epsilon^3)]\mathcal{P} \\
&= \mathcal{P}\mathcal{L}[1 + \epsilon\Sigma_1(t)]\mathcal{P} + \epsilon^2\mathcal{P}\mathcal{L}\Sigma_2(t)\mathcal{P} + \mathcal{O}(\epsilon^3) \\
&= \mathcal{P}\mathcal{L}[1 + \epsilon\Sigma_1(t)]\mathcal{P} + \epsilon^2\mathcal{P}\mathcal{L}_0\Sigma_2(t)\mathcal{P} + \mathcal{O}(\epsilon^3)
\end{aligned}$$

But, due to $[\mathcal{P}, \mathcal{L}_0] = 0$, $\mathcal{P}\mathcal{L}_0\Sigma_2(t) = 0$ and that's why

$$\mathcal{P}\mathcal{L}[1-\Sigma(t)]^{-1}\mathcal{P} = \mathcal{P}\mathcal{L}[1 + \epsilon\Sigma_1(t)]\mathcal{P} + \mathcal{O}(\epsilon^3).$$

To be precise on the limit $t \to \infty$

Up to the first order of ϵ, the limit $\lim_{t\to\infty} \Sigma(t)$ exists. But including higher orders (third and higher), this limit diverges, while $\lim_{t\to\infty}[1 - \Sigma(t)]^{-1}\mathcal{P}$ exists.

Specifying the action of the projector $\mathcal{P}$

In the example:

$\mathcal{H}_B$ **arbitrary system**, $\dim(\mathcal{H}_B) = N$, $\dim(\mathcal{T}(\mathcal{H}_B)) = N^2$,
$\{|b_m\rangle\}_{m=1}^N$ orthonormal basis.

$$X^R_{s=m,n} = \bar{\rho}_A \otimes |b_m\rangle\langle b_n| \quad X^L_{s=m,n} = I_A \otimes |b_m\rangle\langle b_n| \quad \langle\langle X^L_s | X^R_r \rangle\rangle = \delta_{sr}$$

$$\mathcal{P}\rho = \sum_{m,n=1}^{N} \langle\langle X^L_{s=m,n}|\rho\rangle\rangle \, X^R_{s=m,n} = \sum_{m,n=1}^{N} \bar{\rho}_A \otimes \underbrace{\Pi_m}_{|b_m\rangle\langle b_m|} \mathrm{tr}_A(\rho)\Pi_n$$

$$= \bar{\rho}_A \otimes \underbrace{\mathrm{tr}_A(\rho)}_{\rho_B}$$

ρ_B parametrizes the invariant subspace.

Explicit computation for the example

We remark: $\mathcal{P}\mathcal{L}_0\rho = \bar{\rho}_A \otimes \mathrm{tr}_A(\mathcal{L}_A \otimes \mathcal{I}_B \rho) = 0 \implies \mathcal{P}\mathcal{L}_0 = 0$

Therefore we obtain the projected time evolution as:

$$\frac{d}{dt}\mathcal{P}\rho(t) = \mathcal{P}(\mathcal{L}_0 + \epsilon\mathcal{L}_1)\mathcal{P}^{(\epsilon)}\rho(t) =$$

$$\epsilon\mathcal{P}\mathcal{L}_1\mathcal{P}^{(\epsilon)}\rho(t) \sim_{\epsilon^2}$$

$$\epsilon\mathcal{P}\mathcal{L}_1\mathcal{P}\rho(t) + \epsilon^2 \int_0^{t-t_0} ds\, e^{\mathcal{L}_0 s}\mathcal{Q}\mathcal{L}_1\mathcal{P}\rho(t)$$

Lindblad AE: Order by order expansion

$$\mathcal{K}(\mathcal{L}_s) = \mathcal{L}(\mathcal{K})$$

Substituting the asymptotic expansions of the maps

$$\mathcal{L}_s(\rho_s) = \sum_{n=0}^{\infty} \epsilon^n \mathcal{L}_{s,n}(\rho_s), \quad \mathcal{K}(\rho_s) = \sum_{n=0}^{\infty} \epsilon^n \mathcal{K}_n(\rho_s)$$

We obtain at each order in ϵ a different invariance condition.

For $\epsilon = 0$ we need to compute the steady state $\bar{\rho}$ solution of $\mathcal{L}_0(\rho) = 0$.

Sketch of the proof of the main result (1)

We choose $\mathcal{P}$ as projection on invariant manifold wrt $\mathcal{L}_0$.

$$\mathcal{P}(t) = [1 - \Sigma(t)]^{-1}\mathcal{P}$$
$$\Sigma(t) = \int_0^t d\tau \, e^{\mathcal{QLQ}\tau}\mathcal{QLP}e^{-\mathcal{L}\tau}.$$

- We obtain the time integral of $\Sigma(t)$
- We recast the expression so that only terms with $e^{\mathcal{L}}$ appear (we get rid of stuff like $e^{\mathcal{QLQ}t}$)

$$\rightarrow \mathcal{P}^{(\epsilon)}(t) = [1 - \Sigma^{(\epsilon)}(t)]^{-1}\mathcal{P}$$

Sketch of the proof of the main result (2)

$$(\mathcal{L}_0 - \lambda_i^{(\epsilon=0)})\,|r_i^{(\epsilon=0)}\rangle\!\rangle = 0, \quad \langle\!\langle l_i^{(\epsilon=0)}|\,(\mathcal{L}_0 - \lambda_i^{(\epsilon=0)}) = 0,$$

$$(\mathcal{L} - \lambda_i^{(\epsilon)})\,|r_i^{(\epsilon)}\rangle\!\rangle = 0, \quad \langle\!\langle l_i^{(\epsilon)}|\,(\mathcal{L} - \lambda_i^{(\epsilon)}) = 0.$$

In AE, we assume that the invariant subspaces with $\epsilon > 0$ and $\epsilon \to 0$ are smoothly connected: the gap in the spectrum stays large enough for $\epsilon > 0$: $\Delta^{(\epsilon)} = \min_{f,s} \mathrm{Re}(\lambda_f^{(\epsilon)} - \lambda_s^{(\epsilon)})$.

Then, when $t \gg [\Delta^{(\epsilon)}]^{-1}$, the trajectory is restricted to the invariant subspace spanned by $|r_i^{(\epsilon)}\rangle\!\rangle = |r_i^{(\epsilon=0)}\rangle\!\rangle + |dr_i\rangle\!\rangle$.

We define the matrices M and N as $M_{ss'} = \langle\!\langle l_s^{(\epsilon)}|r_{s'}^{(\epsilon=0)}\rangle\!\rangle$ and $N_{ss'} = \langle\!\langle l_s^{(\epsilon=0)}|r_{s'}^{(\epsilon)}\rangle\!\rangle$.

Sketch of the proof of the main result (3)

With all of these definitions we can rewrite $\mathcal{P}(t)$ as:

$$\mathcal{P}^{(\epsilon)}(t) = \sum_{s_a, s_b} e^{\mathcal{L}t} \, |r_{s_a}^{(\epsilon=0)}\rangle\!\rangle \, [A(t)]^{-1}_{s_a s_b} \, \langle\!\langle l_{s_b}^{(\epsilon=0)}| \, ,$$

$$A(t)_{s_a s_b} = \langle\!\langle l_{s_a}^{(\epsilon=0)}| \, e^{\mathcal{L}t} \, |r_{s_b}^{(\epsilon=0)}\rangle\!\rangle \, .$$

- We substitute $e^{\mathcal{L}t}$ with its spectral decomposition in $A(t)$
- We express $[A(t)]^{-1}$ in terms of the M, N matrices previously defined, and we get some decay rates for the fast components.

$$\dots \quad \rightarrow \quad \mathcal{P}^{(\epsilon)} = \lim_{t \to \infty} \mathcal{P}(t) = \sum_{ss'} |r_s^{(\epsilon)}\rangle\!\rangle \, [N^{-1}]_{ss'} \, \langle\!\langle l_{s'}^{(\epsilon=0)}| \, .$$

$\mathcal{P}^{(\epsilon)}$ is a projection. $\mathcal{P}^{(\epsilon)} \, |r_s^{(\epsilon)}\rangle\!\rangle = |r_s^{(\epsilon)}\rangle\!\rangle$: image $\rightarrow$ invariant subspace $\epsilon > 0$. $\mathcal{P}^{(\epsilon)} \, |r_f^{(\epsilon=0)}\rangle\!\rangle = 0$: kernel $\rightarrow$ subspace spanned by the fast relaxing modes with $\epsilon = 0$.